AN EXISTENCE THEOREM FOR A POLYNOMIAL WITH A GIVEN SEQUENCE OF EXTREMA

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Abstract

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MATHEMATICS

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AN EXISTENCE THEOREM FOR A POLYNOMIAL WITH A GIVEN SEQUENCE OF EXTREMA

(Presented by Academician S. N. Bernstein, 18 I 1966)

1. Let the functions $\{\varphi_k(x)\}_{k=0}^n$, continuous on $[a,b]$, form a Chebyshev system (a T_n-system), and let real numbers $v_0, v_1, \dots, v_n$ be given. Denote by V_n the set of all polynomials in the system $\{\varphi_k(x)\}_{k=0}^n$ satisfying the conditions

$$P(x_i) = v_i \qquad (i = 0, 1, \dots, n), \tag{1}$$

where $x_0 = a$, $x_n = b$, and $x_1 < \dots < x_{n-1}$ are arbitrary interpolation nodes from the interval (a,b).

Theorem 1. *Let points be given*

$$a = x_0 < x_1 < \dots < x_{i-1} < x_{i+r} < \dots < x_{n-1} < x_n = b, \tag{2}$$

where i and r are given natural numbers, $i + r \le n$. If the inequalities

$$v_{i+2j-1} > v_{i+2j}, \qquad v_{i+2j+1} > v_{i+2j}$$

$$(j = 0, 1, \dots, m-1); \qquad m = [(r+1)/2], \tag{3}$$

hold, and, for even r, in addition, the inequality

$$v_{i+2m-1} > v_{i+2m}, \tag{4}$$

holds, then in the interval (x_{i-1}, x_{i+r}) there exist points

$$x_i < x_{i+1} < \dots < x_{i+r-1} \tag{5}$$

and a polynomial $P(x) \in V_n$ with nodes (2) and (5) such that

$$\min_{x_{i+2j-1} \le x \le x_{i+2j+1}} P(x) = P(x_{i+2j}) = v_{i+2j},$$

$$\max_{x_{i+2j-2} \le x \le x_{i+2j}} P(x) = P(x_{i+2j-1}) = v_{i+2j-1} \tag{6}$$

for $j = 0, 1, \dots, m-1$, and, for even r, in addition,

$$\max_{x_{i+2m-2} \le x \le x_{i+2m}} P(x) = P(x_{i+2m-1}) = v_{i+2m-1}. \tag{7}$$

Moreover, the polynomial $P(x) \in V_n$ is uniquely determined by the conditions (6), (7) in the sense that if in them the points (5) are replaced by other points $y_i < \dots < y_{i+r-1}$ from (x_{i-1}, x_{i+r}), then there does not exist another polynomial $Q(x) \in V_n$ with nodes (2) and y_s, satisfying the conditions (6), (7).

Theorem 1 is a strengthening of the results of the papers ([1],[2]), in which it is assumed, first, that the functions $\{\varphi_k(x)\}_{k=0}^n$ not only form a Chebyshev system, but are also continuously differentiable and such that the derivative of any polynomial in $\{\varphi_k(x)\}_{k=0}^n$ can have $\le n$ zeros, and, second, that the numbers $v_0, v_1, \dots, v_n$ have successively opposite signs. For $r = n-1$ the question was first considered in the algebraic case ($\varphi_k(x) = x^k$) in a paper of Davis ([3]).

A polynomial in the Chebyshev system $\{\varphi_k(x)\}_{k=0}^n$ which at certain $n+1$ points of the interval $[a,b]$ attains its greatest absolute value with successively opposite signs will, following S. N. Bernstein ([4]), be called an oscillating polynomial. If in Theorem 1 we put $v_k = (-1)^k$ ($k = 0, 1, \dots, n$), then we may formulate the following

Corollary. *For every T_n-system $\{\varphi_k(x)\}_{k=0}^n$ on $[a,b]$ there exists a unique oscillating polynomial $P(x)$, normalized by the condition $P(a) = 1$.*

Moreover, this corollary can be derived directly from the works of S. N. Bernstein ([4]), § 12 and ([5]).

For the proof of Theorem 1 we apply the method of successive approximations ([1],[2]) and induction with respect to the number r. Fix i and put $r = 1$; in this case inequalities (3) will be $v_{i-1} > v_i$, $v_{i+1} > v_i$. Adjoin to the points (2) an arbitrary point y_i from (x_{i-1}, x_{i+1}) and construct a polynomial $A(x)$ from the conditions $A(x_k) = v_k$ ($k \ne i$), $A(y_i) = v_i$. If

$$\min_{x_{i-1} \le x \le x_{i+1}} A(x) = A(y_i) = v_i, \tag{8}$$

then y_i is the desired point, and $A(x)$ the desired polynomial. If, however,

$$\min_{x_{i-1} \le x \le x_{i+1}} A(x) < A(y_i) = v_i, \tag{9}$$

then construct a polynomial $B(x)$ from the conditions $B(x_k) = 0$ $(k \ne i)$ and consider the polynomial $C(x; \lambda) = A(x) + \lambda B(x)$. Since $B(x) \ne 0$ for $x_{i-1} < x < x_{i+1}$, it is possible to choose such a value $\lambda = \lambda_1$ that

$$\min_{x_{i-1} \le x \le x_{i+1}} C(x; \lambda_1) > v_i. \tag{10}$$

Since $\min C(x; \lambda)$ on $[x_{i-1}, x_{i+1}]$ is a continuous function of λ, and $C(x; 0) = A(x)$, it follows from (9) and (10) that there is a $\lambda = \lambda_0$, $0 < \lambda_0 < \lambda_1$, such that

$$\min_{x_{i-1} \le x \le x_{i+1}} C(x; \lambda_0) = v_i.$$

Denote the point at which the minimum of $C(x; \lambda_0)$ is attained by x_i, and $P(x) = C(x; \lambda_0)$ is a polynomial satisfying the conditions of the theorem. Suppose that there exists another polynomial $Q(x)$ such that $Q(x_k) = v_k$ $(k \ne i)$,

$$\min_{x_{i-1} \le x \le x_{i+1}} Q(x) = Q(\xi_i) = v_i, \quad x_i < \xi_i < x_{i+1}.$$

From $P(x_i) = v_i < Q(x_i)$ and $P(\xi_i) > v_i = Q(\xi_i)$ it follows that the polynomial $P(x) - Q(x)$ has a zero in (x_i, ξ_i); in addition, it has n other zeros x_k $(k \ne i)$, which is impossible.

Assume, by induction, that Theorem 1 has been proved for some number r $(1 \le r \le n-2)$ for any i, and show that it is true for the number $r+1$. For definiteness put $r = 2m-1$. Let the inequalities (3), (4) be satisfied and let the points be given

$$a = x_0 < x_1 < \dots < x_{i-1} < x_{i+2m} < \dots < x_n = b. \tag{11}$$

Adjoin to them an arbitrary point x'_{i+2m-1}, $x_{i-1} < x'_{i+2m-1} < x_{i+2m}$. By the induction hypothesis, in (x_{i-1}, x'_{i+2m-1}) there will be points

$$x'_i < \dots < x'_{i+2m-2}$$

and a uniquely determined polynomial $P_1(x) \in V_n$ with nodes x_k, x'_s, which satisfies conditions (7) at the points x'_s $(s = i, \dots, i+2m-2)$. If

$$\max_{x'_{i+2m-2} \le x \le x_{i+2m}} P_1(x) = P_1(x'_{i+2m-1}) = v_{i+2m-1}, \tag{12}$$

then $P_1(x)$ is the desired polynomial. If, however, (12) is not satisfied, then at certain points y of the interval (x'_{i+2m-2}, x_{i+2m}) the equality

$$P_1(y) = v_{i+2m-1}$$

holds. Since $P_1(x)$ is a polynomial in a T_n-system, the number of these points is finite; denote by y'_{i+2m-1} the nearest of them to the point x'_{i+2m-1}, and put, for definiteness, $x'_{i+2m-1} < y'_{i+2m-1}$. Choose

$x''_{i+2m-1} = 0.5\,(x'_{i+2m-1} + y'_{i+2m-1})$, adjoin it to the points (11) and repeat the preceding arguments. In the interval (x_{i-1}, x''_{i+2m-1}) there will be points $x''_i < x''_{i+1} < \dots < x''_{i+2m-2}$ and a polynomial $P_2(x) \in V_n$ with nodes x_k, x''_s, which satisfies conditions (7) at the points x''_s $(s = i, \dots, i+2m-2)$. If $P_2(x)$ does not satisfy the equality

$$\max_{x''_{i+2m-2} \le x \le x_{i+2m}} P_2(x) = P_2(x''_{i+2m-1}) = v_{i+2m-1}, \tag{13}$$

then one must continue the process of constructing the points $x_s^{(k)}$ and the polynomials $P_k(x) \in V_n$ according to the same scheme. Suppose that the process does not terminate at any finite step, and let us prove its convergence.

Thus, let $P_2(x)$ fail to satisfy (13). We shall show that

$$P_2(x) < P_1(x) \quad \text{for } x \in I = (x'_{i+2m-1}, y'_{i+2m-1}). \tag{14}$$

Suppose that $P_1(\xi_{i+2m-1}) = P_2(\xi_{i+2m-1}) = v^*_{i+2m-1}$, $\xi_{i+2m-1} \in I$. Adjoin ξ_{i+2m-1} to (11) and denote by V_n^* the set of polynomials obtained from V_n by replacing v_{i+2m-1} by v^*_{i+2m-1}. Then $P_1(x) \in V_n^*$, $P_2(x) \in V_n^*$, and both polynomials satisfy conditions (6) at x'_s and x''_s $(s = i, \dots, i+2m-2)$, respectively. This contradicts the inductive assumption on the uniqueness of such a polynomial. Hence $P_1(x) \ne P_2(x)$ for $x \in I$. On the other hand, the point y'_{i+2m-1} was chosen so that $P_1(x) > v_{i+2m-1}$ for $x \in I$, and, in particular, $P_1(x''_{i+2m-1}) > P_2(x''_{i+2m-1}) = v_{i+2m-1}$. Similarly to the preceding, we find in (x''_{i+2m-2}, x_{i+2m}) such a point y''_{i+2m-1} that $P_2(y''_{i+2m-1}) = v_{i+2m-1}$. By (14), $y''_{i+2m-1} \in I$ and

$$|y''_{i+2m-1} - x''_{i+2m-1}| < 0.5\,|y'_{i+2m-1} - x'_{i+2m-1}|. \tag{15}$$

We prove that the inequalities

$$x'_i \le x''_i, \dots, x'_{i+2m-2} \le x''_{i+2m-2}. \tag{16}$$

hold. Denote by z the leftmost of the points of I at which $P_2(x) = v_{i+2m-1}$. Choose a point q_{i+2m-1} from (x'_{i+2m-1}, z), adjoin it to (11), and construct a polynomial $Q(x) \in V_n$ which satisfies conditions (6) at the points $q_i < \dots < q_{i+2m-2}$ from (x_{i-1}, q_{i+2m-1}). Arguing as in the derivation of (14), we obtain $P_2(x) < Q(x) < P_1(x)$ for $q_{i+2m-1} < x < z$. It follows that, as $q_{i+2m-1} \to x'_{i+2m-1}$, the polynomial $Q(x)$ tends uniformly to $P_1(x)$ on $[x'_{i+2m-1} + \varepsilon, z]$, $\varepsilon > 0$, and hence on the whole $[a, b]$. Clearly, $Q(x)$ depends continuously on

q_{i+2m-1}. Choose $\delta > 0$ so small that the δ-neighborhoods of the points x'_s $(s = i, \dots, i+2m-2)$ do not intersect, and then choose $q_{i+2m-1} \in I$ so close to x'_{i+2m-1} that the inequalities $|x'_i - q_i| < \delta, \dots, |x'_{i+2m-2} - q_{i+2m-2}| < \delta$ hold. In each δ-neighborhood there lies at least one zero of the polynomial $Q(x) - P_1(x)$; moreover, its zeros include the points (11). Consequently, in each δ-neighborhood there lies exactly one zero. We show that $x'_{i+2m-2} \leq q_{i+2m-2}$. If $x'_{i+2m-2} > q_{i+2m-2}$, then $Q(x) - P_1(x)$ would have at least two zeros in $(q_{i+2m-2}, x'_{i+2m-1})$, because of the inequalities $Q(q_{i+2m-2}) - P_1(q_{i+2m-2}) < 0$, $Q(x'_{i+2m-2}) - P_1(x'_{i+2m-2}) > 0$, $Q(x'_{i+2m-1}) - P_1(x'_{i+2m-1}) < 0$. Suppose that

$$x'_{i+2m-2} = q_{i+2m-2},\ x'_{i+2m-3} = q_{i+2m-3}, \dots,\ x'_{i+k} = q_{i+k},\ x'_{i+k-1} \neq q_{i+k-1}. \tag{17}$$

If $q_{i+k-1} < x'_{i+k-1}$, then $(-1)^{2m-k}\{Q(q_{i+k-1}) - P_1(q_{i+k-1})\} > 0$, $Q(x'_{i+2m-1}) - P_1(x'_{i+2m-1}) > 0$. Consequently, the number of zeros of the polynomial $Q(x) - P_1(x)$ in (q_{i+k-1}, x'_{i+2m-1}) exceeds the number of points (17) by an odd number, and hence the number of zeros in (a, b) is greater than n. Similarly one proves the inequalities $x'_{i+2m-3} \leq q_{i+2m-3}, \dots, x'_i \leq q_i$, from which (16) follows by continuity.

If $\{P_k(x)\} \subset V_n$ is our sequence, then

$$[x^{(k+1)}_{i+2m-1},\, y^{(k+1)}_{i+2m-1}] \subset [x^{(k)}_{i+2m-1},\, y^{(k)}_{i+2m-1}]$$

and inequalities similar to (15) and (16) hold:

$$|y^{(k+1)}_{i+2m-1} - x^{(k+1)}_{i+2m-1}| < 0.5\,|y^{(k)}_{i+2m-1} - x^{(k)}_{i+2m-1}|, \quad x^{(k)}_i \leq x^{(k+1)}_i, \dots, x^{(k)}_{i+2m-2} \leq x^{(k+1)}_{i+2m-2}.$$

It follows that the limits $\lim_{k\to\infty} x^{(k)}_s = x_s$ exist for $s = i, \dots, i+2m-1$. Moreover, $x_s \neq x_{s+1}$; if $x_s = x_{s+1}$, then $\{P_k(x)\}$ would converge to a discontinuous function, which is impossible, since $\{P_k(x)\}$ is uniformly bounded on I. Put $P(x) = \lim_{k\to\infty} P_k(x)$. It is clear that

$$P(x) \in V_n$$

and that it satisfies conditions (6) and (7).

It remains to prove the uniqueness of the polynomial $P(x)$. Suppose that there exists another polynomial $Q(x) \in V_n$ which satisfies conditions (6) and (7) at the points $y_i < \dots < y_{i+2m-1}$. If $x_{i+2m-1} = y_{i+2m-1}$, then there would exist two polynomials $P(x) \in V_n$ and $Q(x) \in V_n$ satisfying conditions (6), which contradicts the inductive assumption on uniqueness. Let, for definiteness, $x_{i+2m-1} < y_{i+2m-1}$. From the preceding it follows that the polynomial $P(x)$ is uniquely determined by the initial choice of the point x'_{i+2m-1}. Therefore we must assume that some points x'_{i+2m-1} from the interval (x_{i+2m-1}, y_{i+2m-1}) lead to the system of points $x_i < \dots < x_{i+2m-1}$ and to the polynomial $P(x)$, while others lead to the system of points $y_i < \dots < y_{i+2m-1}$ and to the polynomial $Q(x)$. But this is impossible, since the points $x_i < \dots < x_{i+2m-2}$ are continuous monotone functions of x'_{i+2m-1}. Indeed, if by ξ_{i+2m-1} we denote the upper bound of those x'_{i+2m-1} which generate the system of points $x_i < \dots < x_{i+2m-1}$, then, by continuity, both the point ξ_{i+2m-1} itself and the point $\xi_{i+2m-1} + \Delta\xi$, for sufficiently small $\Delta\xi > 0$, will lead to the same system of points, which contradicts the definition of ξ_{i+2m-1}. Theorem 1 is proved.

2. A system $\{\varphi_k(x)\}_{k=0}^{n}$ of continuous functions on $[a, b]$, such that $\varphi_k(a) = \varphi_k(b)$, is called a (4) periodic T_n-system on $[a, b]$ if every polynomial with respect to this system has $\le n$ zeros for $a \le x < b$. We note that n must be an even number, since $P(a) = P(b)$ for every polynomial, while there are polynomials that change sign n times. In this case Theorem 1 can be supplemented as follows.

Theorem 2. Let $\{\varphi_k(x)\}_{k=0}^{2m}$ form a periodic T_{2m}-system on $[a, b]$, and let numbers $v_0, v_1, \dots, v_{2m-1}$ be given satisfying the inequalities

$$v_0 > v_1, \quad v_0 > v_{2m-1}, \quad v_{2j} > v_{2j-1}, \quad v_{2j} > v_{2j+1} \quad (j = 1, \dots, m-1).$$

Then there exists a system of points

$$a = x_0 < x_1 < \dots < x_{2m-1} < x_{2m} = b$$

and a uniquely determined polynomial $P(x)$ such that

$$\begin{gathered}\max_{x_0 \le x \le x_1} P(x) = P(x_0) = v_0, \qquad \max_{x_{2m-1} \le x \le x_{2m}} P(x) = P(x_{2m}) = v_0,\\ \max_{x_{2j-1} \le x \le x_{2j+1}} P(x) = P(x_{2j}) = v_{2j}, \qquad \min_{x_{2j} \le x \le x_{2j+2}} P(x) = P(x_{2j+1}) = v_{2j+1}\end{gathered} \tag{18}$$

for $j = 1, \dots, m-1$.

Put $v_0 = v_{2m}$ and consider an arbitrary sequence

$$a < b_1 < b_2 < \cdots$$

converging to b. On the interval $[a, b_k]$, by Theorem 1, one can uniquely construct a polynomial $P_k(x)$ satisfying (18) at certain points

$$a = x_0^{(k)} < x_1^{(k)} < \dots < x_{2m}^{(k)} = b_k.$$

It can be proved that

$$x_s^{(k)} \leq x_s^{(k+1)} \quad (s = 1, \dots, 2m),$$

whence it follows that $\{P_k(x)\}$ converges uniformly to a polynomial $P(x)$ satisfying conditions (18).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.