ON A VARIATIONAL PROBLEM IN HILBERT SPACE AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

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Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV, E. J. ASLANOV

ON A VARIATIONAL PROBLEM IN HILBERT SPACE AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

1. Let H_1 and H_2 be real Hilbert spaces with scalar products $(x, y)_1$ and $(\xi, \eta)_2$; let K be a linear continuous operator acting from H_1 into H_2; and let $E(x)$ be a quadratic functional in H_1, defined by the formula

$$E(x) = (Kx - \xi,\, Kx - \xi)_2 + (x, x)_1. \tag{1}$$

Then the following holds.

Theorem. *For every $\xi \in H_2$ there exists $x_0 \in H_1$ minimizing the functional (1), which is determined uniquely from the equation*

$$x + K^*Kx = K^*\xi. \tag{2}$$

Proof. Obviously, equation (2) has, moreover, a unique solution $x_0 \in H_1$. Therefore, for any $y \in H_1$ we have

$$\begin{aligned} E(x_0 + y) &= E(x_0) + 2(K^*Kx_0 + x_0 - K^*\xi, y)_1 + (Ky, Ky)_2 + (y, y)_1 \\ &= E(x_0) + (Ky, Ky)_2 + (y, y)_1. \end{aligned} \tag{3}$$

It follows from (3) that $E(x_0) < E(x_0 + y)$ for every $y \neq 0$, i.e. $x_0 \in H_1$ is the unique element minimizing the functional.

2. Let H be a real Hilbert space with scalar product (φ, ψ). Then the aggregate of vector functions with values in H, defined on the interval $[0, T]$, with scalar product

$$[f, g] = \int_0^T (f(s), g(s))\, ds,$$

will be a Hilbert space (1), which we shall denote by $L_2 = L_2(H, [0,T])$.

Take as H_1 the set of pairs $\langle f, \varphi\rangle$, $f \in L_2$, $\varphi \in H$, with scalar product of the elements $x = \langle f, \varphi\rangle$ and $y = \langle g, \psi\rangle$, defined by the formula

$$(x, y)_1 = a[f, g] + b(\varphi, \psi), \qquad a > 0,\ b > 0.$$

We define the operator K by the formula

$$Kx = Bf + U\varphi,$$

where B and U are linear continuous operators acting from L_2 into H_2 and from H into H_2, respectively. The operator K^*, acting from H_2 into H_1, will have the form

$$K^*\xi = \langle a^{-1}B^*\xi,\ b^{-1}U^*\xi\rangle.$$

Hence

$$K^*Kx = \langle a^{-1}B^*Bf + a^{-1}B^*U\varphi,\ b^{-1}U^*Bf + b^{-1}U^*U\varphi\rangle,$$

and equation (2) is written in the form of the system

$$\begin{aligned} B^*Bf + af + B^*U\varphi &= B^*\xi, \\ U^*U\varphi + b\varphi + U^*Bf &= U^*\xi, \end{aligned} \tag{4}$$

whose solution minimizes the functional

$$E(f, \varphi) = (Bf + U\varphi - \xi, Bf + U\varphi - \xi)_2 + a\int_0^T \|f(s)\|^2\,ds + b\|\varphi\|^2. \tag{5}$$

3. The constructions given above can be applied to the solution of the following variational problems. Let

$$u(t) = U(t, 0)\varphi + \int_0^t U(t, s)f(s)\,ds$$

be a generalized solution of the Cauchy problem

$$du(t)/dt + A(t)u(t) = f(t), \quad 0 < t \le T, \quad u(0) = \varphi, \tag{6}$$

in the space H, where $A(t)$ is, generally speaking, an unbounded operator in H. Here $U(t, s)$ is an operator-valued function, continuous in the aggregate of variables for $0 \leq s \leq t \leq T$, with values in the ring of linear continuous operators in H (for conditions under which $A(t)$ can generate $U(t,s)$, see ([2–4])).

Problem 1. Find a pair $f \in L_2,\ \varphi \in H$, minimizing the functional

$$E_1(f,\varphi) = \|u(T) - \psi\|^2 + a\int_0^T \|f(s)\|^2\,ds + b\|\varphi\|^2, \quad a > 0,\ b > 0, \tag{7}$$

where $\psi \in H$ is a given element.

The functional E_1 may be regarded as a special case of the functional (5), if one sets $H_2 = H$,

$$Bf = \int_0^T U(T,s)f(s)\,ds, \quad U\varphi = U(T,0)\varphi.$$

Then $U^*\varphi = U^*(T,0)\varphi,\ B^*\varphi = U^*(T,t)\varphi$.

System (4) in this case takes the form

$$af(t) + \int_0^T U^*(T,t)U(T,s)f(s)\,ds + U^*(T,t)U(T,0)\varphi = U^*(T,t)\psi,$$

$$b\varphi + U^*(T,0)U(T,0)\varphi + \int_0^T U^*(T,0)U(T,s)f(s)\,ds = U^*(T,0)\psi. \tag{8}$$

Putting $t = 0$ in the first equation and comparing with the second, we obtain $b\varphi = af(0)$, where $f(t)$ is determined from the loaded integral equation

$$af(t) + \int_0^T U^*(T,t)U(T,s)f(s)\,ds + \frac{a}{b}U^*(T,t)U(T,0)f(0) = U^*(T,t)\psi, \tag{9}$$

Problem 2. Find a pair $f \in L_2,\ \varphi \in H$, minimizing the functional

$$E_2(f,\varphi) = \int_0^T \|U(s)-g(s)\|^2\,ds + a\int_0^T \|f(s)\|^2\,ds + b\|\varphi\|^2, \quad a > 0,\ b > 0, \tag{10}$$

where $g \in L_2$ is a given function.

The functional E_2 may be regarded as a special case of the functional (5), if one sets $H_2 = L_2,\ U\varphi = U(t,0)\varphi$,

$$Bf = \int_0^t U(t,s)f(s)\,ds.$$

Then

$$U^*f = \int_0^T U^*(s,0)f(s)\,ds, \qquad B^*f = \int_t^T U^*(s,t)f(s)\,ds.$$

System (4) in this case takes the form:

$$\int_t^T \int_0^s U^*(s,t)U(s,\tau)f(\tau)\,d\tau\,ds + af(t) + \int_t^T U^*(s,t)U(s,0)\varphi\,ds =$$

$$= \int_t^T U^*(s,t)g(s)\,ds,$$

$$\int_0^T \int_0^s U^*(s,0)U(s,\tau)f(\tau)\,d\tau\,ds + b\varphi + \int_0^T U^*(s,0)U(s,0)\varphi\,ds =$$

$$= \int_0^T U^*(s,0)g(s)\,ds.$$

Just as in problem 1, we obtain $b\varphi = af(0)$, where $f(t)$ is determined from the loaded integral equation

$$af(t) + \int_t^T \int_0^s U^*(s,t)U(s,\tau)f(\tau)\,d\tau\,ds +$$

$$+ \frac{a}{b}\int_t^T U^*(s,0)U(s,0)f(0)\,ds = \int_t^T U^*(s,t)g(s)\,ds. \tag{11}$$

The problems set forth, following Bellman ([5]), may be regarded as problems of optimal control. The first term in the functionals E_1 and E_2 represents the cost of deviation, and the other two terms the cost of control, with a and b determining the weight assigned to the controls f and φ.

4. Many nonstationary problems for partial differential equations can be written in the form (6). Therefore the solution of variational problems 1 and 2 can be applied to the investigation of analogous questions for partial differential equations. As an application, consider the Cauchy problem for the heat equation

$$\partial u(t,x)/\partial t - \partial^2 u(t,x)/\partial x^2 = f(t,x), \qquad -\infty < x < \infty, \quad t \geqslant 0,$$

$$u(0,x) = \varphi(x).$$

Here $H = L_2(-\infty,\infty)$, $A(t)$ does not depend on t and is determined by the equality $Au = -\partial^2 u/\partial x^2$, $D(A) = W_2^2(-\infty,\infty)$. Then A is a self-adjoint positive operator in $L_2(-\infty,\infty)$, and $U^*(t,s) = U(t,s) = \exp(-(t-s)A)$, where (see (6))

$$\exp(-tA)\varphi = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\eta)^2}{4t}\right) \varphi(\eta)\, d\eta, \qquad t > 0.$$

The functionals E_1 and E_2 take, for this case, the form

$$E_1(f,\varphi) = \int_{-\infty}^{\infty} |u(T,x) - \psi(x)|^2\, dx + a\int_0^T \int_{-\infty}^{\infty} |f(t,x)|^2\, dx\, dt + b\int_{-\infty}^{\infty} |\varphi(x)|^2\, dx,$$

$$E_2(f,\varphi) = \int_0^T \int_{-\infty}^{\infty} |u(t,x) - g(t,x)|^2\, dx\, dt + a\int_0^T \int_{-\infty}^{\infty} |f(t,x)|^2\, dx\, dt + b\int_{-\infty}^{\infty} |\varphi(x)|^2\, dx.$$

Problem 1 becomes the problem of the best approximation to a prescribed temperature $\psi(x)$ over a definite time interval T, and its solution is found from the nonhomogeneous integral equation

$$af(t,x) + \int_0^T \int_{-\infty}^{\infty} K(x-\eta,t,s) f(s,\eta)\, d\eta\, ds +$$

$$+ \frac{a}{b} \int_{-\infty}^{\infty} K(x-\eta,t,0) f(0,\eta)\, d\eta = G(t,x), \tag{12}$$

$$K(\xi,t,s) = \frac{1}{2\sqrt{\pi(2T-s-t)}} \exp\left(-\frac{\xi^2}{4(2T-s-t)}\right);$$

$$G(t,x) = \int_{-\infty}^{\infty} \frac{\exp\left(-(x-\eta)^2/4(T-t)\right)}{2\sqrt{\pi(T-t)}}\, \psi(\eta)\, d\eta.$$

Problem 2 becomes the problem of the best approximation to a prescribed temperature regime $g(t,x)$ over a definite time interval T, and its solution is found from equation (12), where

$$K(\xi,t,s)=\sqrt{\frac{2T-s-t}{\pi}}\exp\left(-\frac{\xi^2}{4(2T-s-t)}\right)-\sqrt{\frac{|t-s|}{\pi}}\times$$

$$\times\exp\left(-\frac{\xi^2}{4|t-s|}\right)-\frac{\xi}{4}\left[\Phi\left(\frac{\xi}{2\sqrt{2T-s-t}}\right)-\Phi\left(\frac{\xi}{2\sqrt{|t-s|}}\right)\right];$$

$$G(x,t)=\int_t^T\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\eta)^2}{4(s-t)}\right)}{2\sqrt{\pi(s-t)}}\,g(s,\eta)\,d\eta\,ds.$$

Here

$$\Phi(a)=2\pi^{-1/2}\int_0^a e^{-\tau^2}\,d\tau$$

is the probability integral.

The present article is, in content, close to the works ([7],[8]). Part of the results presented, concerning the problem of the best approximation to a prescribed temperature regime, is contained in ([9],[10]).

Institute of Cybernetics
Academy of Sciences of the Azerbaijan SSR

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Note: Figure translations are in progress. See original paper for figures.