ON VECTOR COUNTABLY ADDITIVE MEASURES

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Abstract

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MATHEMATICS

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ON VECTOR COUNTABLY ADDITIVE MEASURES

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1. Introduction. Consider: 1) the class $\mathcal{P}$ of all subsets of a set U with the usual ring structure, order, and *so*-topology S (([1], § 2); 2) a ring $\mathcal{R}$ in $\mathcal{P}$; 3) the system $\mathbf{K} = \mathbf{K}(\mathcal{R})$ ($\mathbf{D} = \mathbf{D}(\mathcal{R})$) of all finite (countable) disjoint parts of $\mathcal{R}$ ($\mathbf{K} \subseteq \mathbf{D}$); 4) a complete separable real locally convex space G.

Definition 1. We shall call a mapping a of the ring $\mathcal{R}$ into G **countably additive in** $\mathcal{R}$ if, for every $\mathcal{D} \in \mathbf{D}$ for which $\sum \mathcal{D} \in \mathcal{R}$, the family $a\mathcal{D}$ is summable and $\sum a\mathcal{D} = a \sum \mathcal{D}$.

Denote by $A = A(\mathcal{R}, G)$ the set of all mappings of $\mathcal{R}$ into G that are countably additive in $\mathcal{R}$.

Definition 2. We shall call a mapping $a \in A$ **countably additive** if, for every $\mathcal{D} \in \mathbf{D}$, the family $a\mathcal{D}$ is summable.

Denote by $C = C(\mathcal{R}, G)$ the set of all $a \in A$ that are countably additive mappings.

Denote by Γ the set of seminorms on G defining the topology in G. For each $\gamma \in \Gamma$, $a \in A$, $X \in \mathcal{R}$, define

$$v_{\gamma a} X = \sup\{\sum \gamma a \mathcal{K} : \mathcal{K} \in \mathbf{K},\ \sum \mathcal{K} \subseteq X\};$$

we shall call the mappings $v_{\gamma a} : X \to v_{\gamma a} X$ the **variations** for a.

Denote by $V = V(\mathcal{R}, G)$ the set of all $a \in A$ having **bounded variations**; $S = S(\mathcal{R}, G)$—those that are continuous mappings of $\mathcal{R}$ into G; $B = B(\mathcal{R}, G)$—those that are bounded mappings of $\mathcal{R}$ into G; $E = E(\mathcal{R}, G)$—those having an extension $\bar{a} \in A(\bar{\mathcal{R}}, G)$ ($\bar{\mathcal{R}}$ is the closure of the ring $\mathcal{R}$ in $\mathcal{P}$).

The note is devoted to describing the connection between the distinguished classes of countably additive mappings. The main result is expressed as follows:

Theorem 1. $V \subseteq S(E) \subseteq C \subseteq B$.

2. The topology V_a. For each $a \in V$, $\gamma \in \Gamma$, $K \in \mathcal{K}$ ($\mathcal{K}$ is the class of all finite subsets of the set Γ), $\varepsilon > 0$, define

$$\mathfrak{U}(\gamma, \varepsilon) = \{X \in \mathcal{P} : X \subseteq \sum \mathcal{D}_0 \ (\mathcal{D}_0 \in \mathbf{D}), \ \sum v_{\gamma a} \mathcal{D}_0 < \varepsilon\}; \quad \mathfrak{U}(K, \varepsilon) = \bigcap_{\gamma \in K} \mathfrak{U}(\gamma, \varepsilon).$$

For every $\gamma \in \Gamma$, $a \in V$, the variation $v_{\gamma a} \in B(\mathcal{R}, R)$. Consequently, there exists a topology $V_{\gamma a}$ in $\mathcal{P}$, compatible with the ring structure in $\mathcal{P}$, for which $\{\mathfrak{U}(\gamma, \varepsilon)\}_\varepsilon$ is a fundamental system of neighborhoods of zero (([2], § 1). Consider the topology V_a in $\mathcal{P}$ generated by the union $\bigcup (V_{\gamma a})_\gamma$. The following propositions describe some general properties of V_a.

Proposition 1. The topology V_a is compatible with the ring structure in $\mathcal{P}$, and $\{\mathfrak{U}(K, \varepsilon)\}_{K,\varepsilon}$ is a fundamental system of neighborhoods of zero.

Proposition 2. Every class $\mathcal{D} \in \mathbf{D}(\bar{\mathcal{R}})$ is summable, and its sum is equal to the union $\sum \mathcal{D}$.

Denote by $\bar{\mathcal{R}}_a$ the closure of the ring $\mathcal{R}$ in the topology V_a.

Proposition 3. $\bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}_a$.

Proposition 4. Every $a \in V$ is uniformly continuous (V_a).

3. Scheme of the proof of Theorem 1. Theorem 1 is the union of the following propositions.

Proposition 5. $V \subseteq S$.

For every $\gamma \in \Gamma$, $a \in V$, $v_{\gamma a} \in B(\mathcal{R}, R)$. Hence $v_{\gamma a} \in S(\mathcal{R}, R)$ (([1], § 3, Theorem 3). Hence the continuity of a in $O(S)$ and the continuity of a.

Proposition 6. $S \subseteq C$.

For each $a \in C'$, inductively, using the Cauchy criterion (([3]), Ch. 3, § 4, item 2, Theorem 1), a sequence (D_n) is defined such that $\{D_n\} \in \mathbf{D}$ and $(aD_n) \to 0$.

Proposition 7. $V \subseteq E$.

For each $a \in V$ there exists a uniformly continuous extension $\bar{a}$ (Proposition 4; ([3]), Ch. 2, § 3, item 4, Theorem 1). $\bar{a}$ is finitely additive (cf. ([2]), § 2, Lemma 5). For each $\mathcal{D} \in \mathbf{D}(\mathcal{R})$, by Proposition 2, $\sum \mathcal{D} \subseteq \bar{\mathcal{F}}_a$. Consequently, $\bar{a} \in \mathbf{A}(\mathcal{R}_a, G)$ and $a \in E$ (Proposition 3).

Proposition 8. $E \subseteq C$.

This follows directly from the definitions.

Proposition 9. $C \subseteq B$.

This follows from the particular case for numerical mappings; Proposition 5, item 5, § 4, Ch. 3 ([3]) and the corollary of Theorem 3, item 4, § 2, Ch. 4 ([4]).

4. Some corollaries

Corollary 1. *If G is finite-dimensional, then $V = E = S = C = B$.*

It is enough to prove that $B \subseteq V$ and to consider the case $G = R^n$ (Theorem 1; ([4]), Ch. 1, § 2, item 3, Theorem 2). The proof reduces to estimating the variation with the aid of inequality (1), item 1, § 3, Ch. 7 ([5]).

Corollary 2. *If G is Montel, then $C = B$.*

It is enough to prove that $B \subseteq C$ (Theorem 1). For every $a \in B$, $\mathcal{D} \in \mathbf{D}$, the family $a\mathcal{D}$ is summable in the weakened topology (([5]), Ch. 4, § 6, Theorem 1, corollary; § 7, item 2, Theorem 3, corollary) and, consequently, summable (([4]), Ch. 4, § 3, item 4, Proposition 6).

Corollary 3. *If G is a weakly sequentially complete Banach space, then $C = B$.*

It is enough to prove that $B \subseteq C$. For every $a \in B$, $\mathcal{D} \in \mathbf{D}$, the family $a\mathcal{D}$ is summable in the weakened topology. Every sequence (aD_n), for which $\{aD_n\} = a\mathcal{D}$, converges by subsequences in the weakened topology (([3]), Ch. 3, § 4, item 7, Proposition 9; ([6]), Ch. 4, § 1, item 1d) and, consequently, converges by subsequences (([6]), Ch. 4, § 1, Theorem 1). Consequently, $a\mathcal{D}$ is summable (([3]), Ch. 3, § 4, item 7, Proposition 9; ([6]), Ch. 4, § 1, item 1c).

Corollary 4. *If $\mathcal{R}$ is closed (S), then $C = B = A$.*

This follows directly from the definitions, Theorem 1 and the corollary to Proposition 5, § 2, ([1]).

5. Remarks

Remark 1. There exist $\mathcal{R}, G$ such that $V \neq C$.

Let U be the set of all natural numbers, $\mathcal{R}$ the class of all subsets of the set U, and G an infinite-dimensional Banach space. Consider the mapping $a' : n \to a'n$, where $|a'n| = 1/n$ and $(a'n)_{n\in U}$ is summable (([6]), Ch. 4, § 1, Theorem 2). For each $X \in \mathcal{R}$ define

$$aX = \sum (a'n)_{n\in X} \quad ((^3),\ \text{Ch. } 3,\ \S\ 4,\ \text{item } 3,\ \text{Proposition } 2).$$

Then $a \in C,\ a \notin V$.

Remark 2. There exist $\mathcal{R}, G$ such that $C \neq B$.

The following example was proposed by S. V. Nagaev. Let U be the set of natural numbers, $\mathcal{R}$ the class of all finite subsets of the set U, and G the Banach space of all continuous functions on the interval $[0, 1]$. Consider the mapping a': $a'n = \{X \to X^n - X^{n+1}\}$. For each $X \in \mathcal{R}$ define

$$aX = \sum (a'n)_{n\in X};$$

then $a \in B,\ a \notin C$.

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Note: Figure translations are in progress. See original paper for figures.