QUASILINEAR THEORY OF TWO-BEAM INSTABILITY

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Abstract

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QUASILINEAR THEORY OF TWO-BEAM INSTABILITY

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1. The sole criterion for the applicability of the quasilinear approximation in the theory of a rarefied plasma is, as is known, the smallness of the energy of the collective motions of the plasma in comparison with the energy of the individual motion of charged particles. However, the resulting set of equations is still a complicated nonlinear system, and to simplify it one usually proceeds as follows. From the entire plasma a small group of particles, called resonant particles, is singled out, and their interaction with the plasma waves is considered. The number of resonant particles is precisely the small parameter by means of which a simplified system of quasilinear equations is obtained for the distribution function of the resonant particles and the spectral density of suprathermal noise (^{1}).

However, when considering the instability of monoenergetic beams, the indicated separation of resonant particles cannot be carried out, and one must start from the exact system of quasilinear equations, of course under the condition of its applicability.

In the present work a simple physical model is considered, having a definite relation to beam instability, for which the exact quasilinear equations turn out to be sufficiently simple.

2. Thus, let us consider an electron plasma whose behavior is described by the following system of Vlasov equations:

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} - \frac{e}{m}E\frac{\partial F}{\partial v} = \frac{1}{2}\frac{n_0}{\tau}\left[\delta(v - v_0) + \delta(v + v_0)\right] - \frac{F}{\tau}, \tag{1}$$

$$\frac{\partial E}{\partial x} = -4\pi e\left[\int_{-\infty}^{\infty} F\,dv - n_0\right], \tag{2}$$

where n_0 is the ion density, F is the electron distribution function. The first term on the right-hand side of equation (1) describes a source, as a result of whose action n_0/τ electrons appear per unit volume per unit time, half of them moving with velocity v_0, and the other half with velocity $-v_0$. The second term describes absorption of particles, whose characteristic time is the quantity τ. The exact quasilinear equations have the form:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\left\langle E\frac{\partial F'}{\partial v}\right\rangle = \frac{1}{2}\frac{n_0}{\tau}\left[\delta(v - v_0) + \delta(v + v_0)\right] - \frac{f}{\tau}, \tag{3}$$

$$\frac{\partial F'}{\partial t} + v\frac{\partial F'}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = -\frac{F'}{\tau}, \tag{4}$$

$$\frac{\partial E}{\partial x} = -4\pi e\int_{-\infty}^{\infty} F'\,dv, \tag{5}$$

where f is the electron distribution function averaged over macroscopic pulsations; F' is the macroscopic pulsations of the electron distribution function; E is the pulsating longitudinal electric field,

and the angle brackets in (3) denote the indicated averaging over macroscopic pulsations. In what follows we shall investigate only a plasma that is stationary and homogeneous on the average. This means that $f = f(v)$, and one may pass to the Fourier representation, first introducing, instead of the field intensity E, the potential φ. Then the equations for the averaged electron distribution function and for the intensity of the potential pulsations are obtained without difficulty:

$$\frac{e^2}{m^2}\int k\,\frac{d}{dv}\left[\frac{1/\tau}{(\omega - kv)^2 + 1/\tau^2}\,k\,\frac{df}{dv}\right] I_{k\omega}\,dk\,d\omega - \frac{f}{\tau} = \frac{1}{2}\frac{n_0}{\tau}\left[\delta(v - v_0) + \delta(v + v_0)\right], \tag{6}$$

$$\left[1 + \frac{4\pi e^2}{mk}\int_{-\infty}^{\infty}\frac{df/dv}{\omega - kv + i/\tau}\,dv\right] I_{k\omega} = 0, \tag{7}$$

where the intensity $I_{k\omega}$ is defined by the relation

$$\langle\varphi_{k\omega}\varphi^*_{k'\omega'}\rangle = I_{k\omega}\delta(k - k')\delta(\omega - \omega')$$

valid in the stationary and homogeneous case, and $\varphi_{k\omega}$ is the Fourier transform of the potential φ.

For the plasma under consideration one can obtain the law of conservation of energy, if equation (6) is multiplied by $mv^2/2$ and integrated over all velocities, using equation (7). This conservation law has the form

$$\frac{1}{4\pi}\int k^2 I_{k\omega}\,dk\,d\omega + \frac{m}{2}\int_{-\infty}^{\infty} v^2 f(v)\,dv = \frac{1}{2}n_0 m v_0^2, \tag{8}$$

where the first term on the left-hand side represents the energy density of the plasma oscillations; it must be small in comparison with the remaining two in order for the quasilinear approximation to be valid. In what follows we shall show that this is indeed the case.

3. Let us now proceed to finding the solution of the system (6)–(7) over the entire range of variation of the parameter τ: $0 \leq \tau < \infty$. It is not difficult to see that the solution of equation (7) is written in the form

$$I_{k\omega} = I\delta(\omega-\omega_0)\delta(k-k_0), \tag{9}$$

where I is an unknown constant, and ω_0 and k_0 are that solution of the complex transcendental equation

$$1 + \frac{4\pi e^2}{mk}\int_{-\infty}^{\infty}\frac{df/dv}{\omega - kv + i/\tau}\,dv = 0, \tag{10}$$

which corresponds to the maximal imaginary part of the frequency, equal to zero.

Substituting (9) into (6) and taking as the four fundamental dimensional quantities the velocity v_0, the time $\omega_e^{-1} = (4\pi e^2 n_0/m)^{-1/2}$, the electron mass m, and the charge e, we obtain the basic equations in the following dimensionless form:

$$\frac{d}{dv}\left[D(v)\frac{df}{dv}\right] - \xi f = -\frac{1}{2}\xi[\delta(v-1)+\delta(v+1)], \tag{11}$$

$$1 + \frac{1}{k}\int_{-\infty}^{\infty}\frac{df/dv}{\omega - kv + i\xi}\,dv = 0, \tag{12}$$

where the "diffusion" coefficient $D(v)$ is equal to

$$D(v) = \frac{\xi k^2 I}{(\omega - kv)^2 + \xi^2}, \tag{13}$$

and the quantity is taken as the single free dimensionless parameter

$$\xi = 1/\omega_e\tau. \tag{14}$$

The law of conservation of energy then takes the form

$$k^2 I + \frac{1}{2}\int_{-\infty}^{\infty} v^2 f(v)\,dv = \frac{1}{2}. \tag{15}$$

It is easy to verify that the solution of the system (11)–(12) for $\xi^2 > \xi_c^2 = 1/8 = 0.125$ is the set

$$f(v) = [\delta(v-1) + \delta(v+1)]/2, \qquad I = 0. \tag{16}$$

The critical value ξ_c is determined by the condition that at $\xi = \xi_c$ there first appears, for the distribution (16), a solution of equation (12) with imaginary part of the frequency equal to zero; moreover $\operatorname{Re}\omega = 0$ and $k^2 = k_c^2 = 3/8 = 0.375$.

For small supercriticality ($\xi_c - \xi \ll \xi_c$), by perturbation theory one can find the intensity of the standing wave with $\omega = 0$ and $k = k_c$: $I = \frac{128}{27}\xi_c(\xi_c - \xi)$.

However, of considerably greater interest is the limiting case $\xi = 0$, when the level of suprathermal noise becomes maximal. In this case, as a result of the "diffusion" of particles on the waves, the electron distribution is broadened, and in this limiting case it will also be maximal.

For $\xi = 0$, equation (11) assumes the especially simple form:

$$\frac{d}{dv}\left[\frac{I}{v^2}\frac{df}{dv}\right] - f = -\frac{1}{2}[\delta(v-1) + \delta(v+1)]. \tag{17}$$

The solution of the inhomogeneous equation (17) is easily expressed in the form of a linear combination of two solutions of the homogeneous equation

$$\frac{d}{dv}\left[\frac{I}{v^2}\frac{df}{dv}\right] - f = 0,$$

each of which has the form

$$f(v) = C v^{3/2} Z_{3/4}(iv^2/2\sqrt{I}),$$

where $Z_{3/4}$ are cylindrical functions of imaginary argument of order 3/4.

From the symmetry of the problem it is clear that the distribution f must be an even function of v, and then the imaginary part of equation (12) vanishes identically for $\omega_0 = \operatorname{Re}\omega = 0$, while the real part takes the form

$$1 - \int_{-\infty}^{\infty} \frac{v}{k^2 v^2 + \gamma^2}\frac{df}{dv}\,dv = 0, \tag{18}$$

where $\gamma = \operatorname{Im}\omega$, and f is the solution of equation (17). We must find the value k for which the maximum of γ, as a function of k, is equal to zero; this can occur only, generally speaking, for a definite value of the parameter $I = I_{\max}$. To find the maximum of the function $\gamma = \gamma(k)$, specified implicitly by equation (18), we solve the system:

$$L(\gamma, k^2) \equiv 1 - \int_{-\infty}^{\infty} \frac{v}{k^2 v^2 + \gamma^2} \frac{df}{dv}\, dv = 0, \tag{19}$$

$$\frac{\partial L}{\partial k^2} = \int_{-\infty}^{\infty} \frac{v^3}{(k^2 v^2 + \gamma^2)^2} \frac{df}{dv}\, dv = 0. \tag{20}$$

(Note that the function f does not depend on the parameter k.) For $\gamma = 0$ the left-hand side of equation (20), substituted into (19), gives $k = 0$; whereas the equation

(20) takes the form

$$\int_0^{\infty} \frac{1}{v} \frac{df}{dv}\, dv = 0, \tag{21}$$

from which the maximum noise level $I_{\max}$ is determined.

The numerical solution of equations (17) and (21) gives for $I_{\max}$ a value less than 0.1, and the distribution function in this limiting case has the form shown in Fig. 1.

The function presented in Fig. 1 does indeed satisfy equation (21). It is interesting to note that this is easily verified if

Fig. 1 Fig. 2

Fig. 1 **Fig. 2**

the function is replaced by a broken line (see Fig. 2) and the integral (21) is calculated:

$$\int_0^{\infty} \frac{1}{v} \frac{df}{dv}\, dv \simeq C \int_{0.5}^{1} \frac{dv}{v} - C \int_{1}^{2} \frac{dv}{v} = C \ln 2 - C \ln 2 = 0.$$

Thus, the smallness found for the field energy density in comparison with the beam energy indicates the applicability of the quasilinear approximation in the problem under consideration. The physical reason for this smallness is connected with the circumstance that the "diffusion" of beam electrons on the standing wave occurs more or less uniformly both toward lower velocities and toward higher ones. The electrons take on the overwhelming part of the beam energy, so that only little energy remains for the field, and the noise level turns out to be low.

4. In an analogous way one could consider the problem of electron beams of different density and the instability of a combination of electron and ion beams.

The theory considered here may prove applicable to certain real problems. As an example we point to the investigation of noise in Penning-type discharges, where electrons arriving from two opposite cathodes can, under certain assumptions, be described by the first term on the right-hand side of equation (1). The departure of electrons to the anode after they have produced ionization in the discharge volume can be described with the aid of the second term on the right-hand side of (1).

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CITED LITERATURE

1. *Problems of Plasma Theory*, ed. M. A. Leontovich, vol. 3, Moscow, 1963; vol. 4, Moscow, 1964.

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