SOLUTION OF THE SPATIAL AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR MULTIPLY CONNECTED BODIES OF REVOLUTION BY MEANS OF GENERALIZED ANALYTIC FUNCTIONS

THEORY OF ELASTICITY

1966

Abstract

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UDC 539.311

THEORY OF ELASTICITY

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SOLUTION OF THE SPATIAL AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR MULTIPLY CONNECTED BODIES OF REVOLUTION BY MEANS OF GENERALIZED ANALYTIC FUNCTIONS

(Presented by Academician S. A. Khristianovich, November 2, 1965)

In papers (^{1,2}) a method was given for solving spatial axisymmetric problems of the theory of elasticity by means of analytic functions of a complex variable. In paper (^{3}), for this purpose, p-analytic functions were used. The author (^{4}), for the case of simply connected bodies, applied generalized analytic functions of a somewhat different class. Below the application of this method to multiply connected bodies is considered.

1. Let D be the plane domain occupied by the meridional section of a body of revolution. The outer boundary L_{n+1} of this domain may consist either of one closed contour intersecting the axis of symmetry oz, or of two closed contours L'_{n+1} (for $r > 0$) and L''_{n+1} (for $r < 0$), situated on both sides of oz. Let L'_k and L''_k $(k = 1, 2, \dots, m)$ be symmetrically situated inner contours not intersecting the axis oz, and let L_k $(k = m+1, \dots, n)$ be the remaining inner contours. By $L = \sum L_k$ we shall understand the complete boundary of the domain. We assume that all contours have curvature satisfying the condition H (Hölder).

It can be shown (^{4}) that the differential equations of equilibrium will be satisfied if the components of the elastic displacement w, u are represented in the form

$$2G(w + iu) = \chi'\Phi(t, \bar{t}) - t\Phi'(t, \bar{t}) - \overline{\Psi(t, \bar{t})} \qquad (1)$$

$$(\chi' = 3.5 - 4\nu, \quad t = z + ir).$$

Here ν is Poisson' s ratio; G is the shear modulus; $\Phi(t, \bar{t})$ and $\Psi(t, \bar{t})$ are generalized analytic functions satisfying the complex equation of the form $2\partial\Phi/\partial\bar{t} - (\Phi - \bar{\Phi})/(t - \bar{t}) = 0$ and the condition $\Phi(\bar{t}, t) = \overline{\Phi(t, \bar{t})}$. By $\Phi'(t, \bar{t})$ is meant the derivative in the sense of Bers with respect to the pair $(1, i/r)$ (^{5}). For the stresses the formulas will be

$$\sigma_z + \sigma_r + \sigma_\theta = 2(1+\nu)(\Phi' + \bar{\Phi}'), \qquad \sigma_z + \sigma_r = 2(\Phi' + \bar{\Phi}') - \frac{2Gu}{r}, \quad (2)$$

$$\sigma_z + i\tau_{zr} = 1.5\Phi' + \bar{\Phi}' - t\bar{\Phi}'' - \bar{\Psi}'.$$

From the conditions of single-valuedness of the stresses and displacements there follow the representations

$$\Phi(t,\bar{t}) = \Phi_*(t,\bar{t}) + \sum_{k=1}^{n} A_k \Theta(t,\bar{t}; t_k, \bar{t}_k) + \sum_{k=1}^{m} B_k \Xi(t,\bar{t}; t_k, \bar{t}_k), \quad (3)$$

$$\Psi(t,\bar{t}) = \Psi_*(t,\bar{t}) - \chi' \sum_{k=1}^{n} A_k \Theta(t,\bar{t}; t_k, \bar{t}_k) + \chi' \sum_{k=1}^{m} B_k \Xi(t,\bar{t}; t_k, \bar{t}_k),$$

where A_k and B_k are real constants; t_k are certain arbitrary points situated inside the contours L'_k $(k = 1, 2, \dots, m)$ and L_k $(k =$

$= m+1, \dots, n)$; $\Phi_*(t,\bar{t})$ and $\Psi_*(t,\bar{t})$ are generalized analytic functions regular in D. The functions

$$\Theta(t,\bar{t}; t_k, \bar{t}_k) = -\frac{1}{|t-\bar{t}|} \int_{\bar{t}}^{t} \sqrt{\frac{\xi - \bar{t}}{\xi - t}} \frac{d\xi}{\sqrt{(\xi - t_k)(\xi - \bar{t}_k)}},$$

$$\Xi(t,\bar{t}; t_k, \bar{t}_k) = -\frac{1}{|t-\bar{t}|} \int_{\bar{t}}^{t} \sqrt{\frac{\xi - \bar{t}}{\xi - t}} \frac{2\xi - t_k - \bar{t}_k}{\sqrt{(\xi - t_k)(\xi - \bar{t}_k)}} d\xi - \pi \quad (4)$$

are analogues of the logarithm: upon traversing the contour L'_k counterclockwise, $\Xi(t,\bar{t}; t_k, \bar{t}_k)$ acquires the increment 2π, while $\Theta(t,\bar{t}; t_k, \bar{t}_k)$ acquires the increment $2\pi/(t-\bar{t})$. These functions can be expressed in terms of elliptic integrals of the first and second kind.

When t coincides with a contour point τ, equality (1) turns into a boundary condition for Φ and Ψ corresponding to the second fundamental problem. For the first fundamental problem one can obtain

$$0.5\,\Phi(\tau,\bar{\tau}) + \tau\Phi'(\tau,\bar{\tau}) + \overline{\Psi(\tau,\bar{\tau})} -$$

$$-2(1-\nu) \int_{\tau_k}^{\tau} \left[\Phi(t,\bar{t}) - \overline{\Phi(t,\bar{t})} - \frac{C'_k}{t-\bar{t}} \right] \frac{d(t+\bar{t})}{t-\bar{t}} -$$

$$-\frac{2(1-\nu)}{\tau-\bar{\tau}}\,C_k' - C_k = -R_k + \frac{i}{y} Z_k \quad \text{on } L_k' \ (k = 1, 2, \dots, n+1), \tag{5}$$

where

$$Z_k = \int_{\tau_k}^{\tau} r p_z \, ds, \qquad R_k = \int_{\tau_k}^{\tau} \left(p_r + \frac{1}{r^2} Z_k \frac{dz}{ds} \right) ds, \qquad y = \operatorname{Im} \tau.$$

Here τ_k denotes fixed points of the contours L_k' (for $k \geq m+1$ they lie on the axis of symmetry); p_z and p_r are the intensities of the external forces acting on the surface formed by rotating the contour L_k' about the oz-axis; C_k and C_k' are real constants, and it is known for them that

$$C_k' = 4\pi \sum_{j=m+1}^{k} A_j \quad (k = m+1, \dots, n), \qquad C_{n+1} = C_{n+1}' = 0 \tag{6}$$

(the contours L_k, $k = m+1, \dots, n$, are numbered in the order in which they intersect the axis of symmetry). The coefficients A_k and B_k must satisfy the system of algebraic equations

$$A_k = \frac{1}{4\pi(1-\nu)} \int_{L_k} r p_z \, ds \quad (k = 1, 2, \dots, n), \tag{7}$$

$$4\pi B_k + \int_{L_k'} \left(\Phi - \overline{\Phi} - \frac{1}{t-\bar{t}} C_k' \right) \frac{d(t+\bar{t})}{t-\bar{t}} = \frac{1}{2(1-\nu)} \int_{L_k'} \left(p_r + \frac{1}{r^2} Z_k \frac{dr}{ds} \right) ds$$

$$(k = 1, 2, \dots, m). \tag{8}$$

It can be shown that the regular solution of both fundamental problems is unique.

2. Let us obtain integral equations for the solution of the boundary-value problems. In the case of the second fundamental problem we represent the regular part of the expression

of equations (3) in the form of generalized Cauchy-type integrals

$$\Phi_*(t, \bar{t}) = \frac{1}{2\pi i} \int_L F(\tau, \bar{\tau}) W(t, \bar{t}; \tau, \bar{\tau}) \, d\tau \quad \left(F(\bar{\tau}, \tau) = \overline{F(\tau, \bar{\tau})} \right),$$

$$\Psi_*(t, \bar{t}) = \frac{1}{2\pi i} \int_L \bar{F} \bar{W} [g \, d\tau + 0.5 \, d\bar{\tau}] - \frac{1}{2\pi i} \int_L F \left[\bar{\tau} \frac{\partial W}{\partial z} \, d\tau - W \, d\bar{\tau} \right], \tag{9}$$

where W is the generalized Cauchy kernel (^{4}), $g = -\chi' + 0.5$ for the second fundamental problem and $g = 1$ for the first fundamental problem.

Introduce the notation (τ_0 is a point of the contour)

$$\Omega(\tau_0) = \frac{g}{2\pi i}\int_L F[W\,d\tau - \bar{W}\,d\bar{\tau}] - \frac{1}{2\pi i}\int_L \bar{F}\bar{W}\left[d\tau - \frac{\tau-\tau_0}{\bar{\tau}-\bar{\tau}_0}\,d\bar{\tau}\right] -$$

$$-\frac{1}{4\pi i}\int_L F\left[1+\frac{\bar{\tau}-\tau_0}{\tau-\bar{\tau}_0}\right][W+\bar{W}]\,d\tau, \qquad W = W(\tau_0,\bar{\tau}_0;\tau,\bar{\tau}). \tag{10}$$

Substitute (3) into (1). Letting t tend to τ_0 and noting that

$$\Omega(\tau_0) = -gF(\tau_0,\bar{\tau}_0) + (g-0.5)\Phi_*^+(\tau_0,\bar{\tau}_0) + +\tau_0\bar{\Phi}_*'^+(\tau_0,\bar{\tau}_0) + \Psi_*^+(\tau_0,\bar{\tau}),$$

where on the right-hand side stand the boundary values of the generalized Cauchy-type integrals (9), we obtain the integral equation for the density $F(\tau,\bar{\tau})$

$$(\chi'-0.5)F(\tau_0,\bar{\tau}_0) - \Omega(\tau_0) - \sum_{j=1}^{n} A_j S_j(\tau_0) - \sum_{j=1}^{m} B_j T_j(\tau_0) =$$

$$= 2G(w+in) \quad \text{on } L, \tag{11}$$

where we put

$$A_j = \operatorname{Re}\int_{L_j'} F(\tau,\bar{\tau})\,ds, \qquad B_j = \operatorname{Im}\int_{L_j'} F(\tau,\bar{\tau})\,ds. \tag{12}$$

By S_j and T_j are denoted known functions containing Θ, Ξ, and their derivatives.

In the case of the first fundamental problem, both expressions (9) should be supplemented by the term

$$\sum_{j=1}^{n} b_j\Theta'(t,\bar{t};t_j,\bar{t}_j) \quad \text{with} \quad b_j = \operatorname{Re}\int_{L_j'} \overline{F(\tau,\bar{\tau})}(\tau-\bar{\tau})\,d\tau.$$

Substituting (3) into (5) and transferring the known terms to the right, we shall have the integral equation

$$F(\tau_0,\bar{\tau}_0) + \Omega(\tau_0) + \sum_{j=1}^{n} b_j R_{kj}(\tau_0) + \sum_{j=1}^{m} B_j T_{kj}(\tau_0) - C_k' P_k(\tau_0) - C_k -$$

$$-(1-\nu)\int_{\tau_k}^{\tau_0}(F-\bar{F})\frac{d(\tau+\bar{\tau})}{\tau-\bar{\tau}}-\frac{1-\nu}{\pi i}\int_L FQ_k\,d\tau=f_k(\tau_0)\quad\text{on } L_k'$$

$$(k=1,2,\dots,n+1).\tag{13}$$

Here R_{kj}, T_{kj}, and P_k are known functions,

$$Q_k(\tau_0,\bar{\tau}_0;\tau,\bar{\tau})=\int_{\tau_k}^{\tau_0}\left[W(t,\bar{t};\tau,\bar{\tau})-W(t,\bar{t};\tau,\bar{\tau})\right]\frac{d(t+\bar{t})}{t-\bar{t}}.\tag{14}$$

The prescribed functions $f_k(\tau_0)$ are continuous on the contours L_k', cut at the points τ_k. For the constants C_k and C_k' we take

$$C_k=-\operatorname{Re}\int_{L_k'}F(\tau,\bar{\tau})\,ds\quad(k=1,2,\dots,n),\qquad C_{n+1}=0,\tag{15}$$

$$C_k'=-\operatorname{Im}\int_{L_k'}F(\tau,\bar{\tau})\,ds\quad(k=1,2,\dots,m),\qquad C_k'=0\quad(k\geq m+1).$$

The coefficients B_j can be eliminated by means of (8), which, however, is not necessary in a practical solution.

The integral equations obtained are analogues of D. I. Sherman' s equations [6] for the plane problem. It is easy to show that they reduce to a system of two real Fredholm equations of the second kind. Repeating the arguments of [6], one can verify the solvability of these equations if their right-hand side is differentiable in the class H.

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Received
28 IV 1965

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