ONE PROPERTY OF A SOLUTION OF THE EQUATION \(u_{xx}u_{yy}-u_{xy}^{2}=1\)

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Abstract

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ONE PROPERTY OF A SOLUTION OF THE EQUATION $u_{xx}u_{yy} - u_{xy}^2 = 1$

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In the present paper, for a solution of the equation $u_{xx}u_{yy} - u_{xy}^2 = 1$, an expression is given for the first-order derivatives on the boundary of a convex domain in terms of a single function of one variable.

Consider the Monge–Ampère equation of elliptic type

$$u_{xx}u_{yy} - u_{xy}^2 = 1 \tag{1}$$

in a finite simply connected domain G, bounded by a convex curve Γ, given in the form

$$x = \varphi(s), \qquad y = \psi(s), \qquad 0 \le s \le s^*. \tag{2}$$

Here s is arc length, s^* is the length of the contour Γ, and as s increases the contour Γ is traversed in the positive direction (the domain G remains on the left).

Let $f(s)$ be a function defined for $0 \le s \le s^*$. We shall assume that: a) the functions $\varphi(s), \psi(s), f(s)$ belong to the class* $C^{k,\lambda}$, $k \ge 4$, $0 < \lambda < 1$; b) at $s = 0$ and $s = s^*$ the values of the functions $\varphi(s), \psi(s), f(s)$ coincide together with their derivatives up to order k inclusive; c) the curvature at all points of Γ is not less than $\chi_0 = \text{const} > 0$.

There exist [1,2] two and only two solutions of equation (1) of class $C^{k,\mu}(G+\Gamma)$, $0 < \mu \le \lambda$, which take on Γ the value $f(s)$

$$u|_\Gamma = f(s). \tag{3}$$

For one of these solutions the functions u_{xx} and u_{yy} are positive in $G + \Gamma$, for the other they are negative.

Consider the solution for which u_{xx}, u_{yy} are positive. Let (x_1, y_1), (x_2, y_2) be two distinct points belonging to $G + \Gamma$; then the inequality ([3])

$$(x_2 - x_1)[u_x(x_2, y_2) - u_x(x_1, y_1)] + (y_2 - y_1)[u_y(x_2, y_2) - u_y(x_1, y_1)] > 0 \quad (4)$$

is valid.

Introduce new variables ξ, η and a new function Φ by the formulas

$$\xi = u_x + x, \qquad \eta = u_y + y, \qquad (x, y) \in G + \Gamma; \quad (5)$$

$$\Phi = xu_x + yu_y - u + (x^2 + y^2)/2. \quad (6)$$

From (4) follows the inequality

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 < (\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2. \quad (7)$$

Here the points (ξ_1, η_1), (ξ_2, η_2) of the ξ, η-plane correspond to the points (x_1, y_1), $(x_2, y_2) \in G + \Gamma$ by virtue of formulas (5).

It follows from inequality (7) that formulas (5) effect a homeomorphic mapping of $G + \Gamma$ onto some domain $D + L$ of the ξ, η-plane. The equation of L—the image of Γ—can be written in the form $(0 \leq s \leq s^*)$

$$\xi_L(s) = u_x(\varphi(s), \psi(s)) + \varphi(s), \qquad \eta_L(s) = u_y(\varphi(s), \psi(s)) + \psi(s). \quad (8)$$

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* The derivative of order k satisfies the Hölder condition with exponent λ.

In view of inequality (4), the first derivatives with respect to s of the functions $\xi_L(s)$ and $\eta_L(s)$ cannot vanish simultaneously. The Jacobian of the mapping (5), in view of (1), is positive in $G + L$; therefore, as s increases the curve L is traversed in the positive direction.

Let l denote the length of an arc of the contour L, and let l^* be the length of L; in so doing, as the origin of measurement we choose the point corresponding to $s = 0$, and we shall assume that, as l increases, the curve L is traversed in the positive direction. From the equation $l = l(s)$, $0 \leq s \leq s^*$, one can determine the function of class $C^{k-1,\mu}$, $s = s(l)$, $0 \leq l \leq l^*$.

The function $\Phi(\xi, \eta)$ belongs to the class $C^{k-1,\mu}(D + L)$ and satisfies in D the equation*

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = 1.$$

At the same time, in $D + L$ the equalities

$$x = \Phi_\xi, \qquad y = \Phi_\eta, \qquad u = \xi\Phi_\xi + \eta\Phi_\eta - \frac{1}{2}(\Phi_\xi^2 - \Phi_\eta^2). \tag{9}$$

hold. Thus, on L the equalities (8) and the equalities

$$\partial\Phi/\partial\xi\big|_L = \varphi(s), \qquad \partial\Phi/\partial\eta\big|_L = \psi(s). \tag{10}$$

are satisfied.

Let the function $w_1(\zeta) = \xi + i\eta$ map conformally the unit disk $K + C : |\zeta| = 1$ of the plane of the complex variable ζ onto the domain $D + L$, and let τ be the polar angle in the ζ-plane. We shall assume that the point determined by the equalities $|\zeta| = 1$, $\tau = 0$ corresponds to the point of the contour L taken as the origin for measuring the arc length l, and that the point $\zeta = 0$ corresponds to the point (ξ_0, η_0), which under the transformation (5) is the image of some point $(x_0, y_0) \in G$. The function (*) $w_1(\zeta) \in C^{k-1,\mu}(K + C)$, and the function $l = l(\tau)$, $0 \le \tau \le 2\pi$, belongs to the class $C^{k-1,\mu}$.

Denote

$$s = s(l(\tau)) \equiv g(\tau), \qquad 0 \le \tau \le 2\pi. \tag{11}$$

The function $g(\tau)$ realizes a homeomorphic mapping of $[0, 2\pi]$ onto $[0, s^*]$ and belongs to the class $C^{k-1,\mu}$.

Consider the function (*) of ζ of class $C^{k-1,\mu}(K + C)$

$$(-2\Phi_\eta + \eta) + i(-2\Phi_\xi + \xi) = w_2(\zeta).$$

In view of equalities (10) and (11), for $0 \le \tau \le 2\pi$ we have

$$\operatorname{Re}\left[\frac{1}{2}w_1(\zeta) + \frac{i}{2}w_2(\zeta)\right]_{\zeta = e^{i\tau}} = \varphi(g(\tau)),$$

$$\operatorname{Re}\left[-\frac{i}{2}w_1(\zeta) - \frac{1}{2}w_2(\zeta)\right]_{\zeta = e^{i\tau}} = \psi(g(\tau)).$$

Hence, using the Schwarz formula, we find (*)

$$w_1(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\varphi(g(\omega)) + i\psi(g(\omega))]\,\frac{e^{i\omega}+\zeta}{e^{i\omega}-\zeta}\,d\omega + a,$$

$$\tag{12}$$

$$w_2(\zeta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}[i\varphi(g(\omega)) + \psi(g(\omega))]\,\frac{e^{i\omega}+\zeta}{e^{i\omega}-\zeta}\,d\omega + b.$$

Here $a = u_x(x_0, y_0) + iu_y(x_0, y_0)$, $b = u_y(x_0, y_0) + iu_x(x_0, y_0)$. Setting $\zeta = e^{it}$ in the first formula (12) and taking account of formulas (11) and (8), we obtain

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* The indices ξ, η denote partial derivatives.

representation of the boundary derivatives of first order of the function $u(x, y)$ in terms of $g(\tau)$

$$\left.\frac{\partial u}{\partial x}\right|_{\Gamma} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(g(\omega))\operatorname{ctg}\frac{\tau-\omega}{2}\,d\omega + u_x(x_0, y_0),$$

$$0 \leqslant \tau \leqslant 2\pi \tag{13}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\Gamma} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(g(\omega))\operatorname{ctg}\frac{\tau-\omega}{2}\,d\omega + u_y(x_0, y_0).$$

Remark 1. As is seen from formulas (3), (11), (13), the function $g(\tau)$ satisfies the singular integral equation ($0 \leqslant \tau \leqslant 2\pi$)

$$f'(g(\tau)) - \varphi'(g(\tau))u_x(x_0, y_0) - \psi'(g(\tau))u_y(x_0, y_0) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[\varphi(g(\omega))\psi'(g(\tau)) - \varphi'(g(\tau))\psi(g(\omega))]\operatorname{ctg}\frac{\tau-\omega}{2}\,d\omega;$$

$$f'(s) = df(s)/ds; \qquad \varphi'(s) = d\varphi(s)/ds, \qquad \psi'(s) = d\psi(s)/ds.$$

Remark 2. Formulas (9) and (12) give a parametric representation of the solution of the Dirichlet problem for equation (1).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.