THE CAUCHY PROBLEM FOR A CLASS OF LINEAR FACTORIZED DIFFERENTIAL-OPERATOR EQUATIONS

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Abstract

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MATHEMATICS

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THE CAUCHY PROBLEM FOR A CLASS OF LINEAR FACTORIZED DIFFERENTIAL-OPERATOR EQUATIONS

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1. Let

$$T_t = D^p + a_1(t)D^{p-1} + \dots + a_p(t), \qquad D = \partial/\partial t, \qquad p \geqslant 1,$$

and let X be a linear operator, independent of t, acting in the space of the variables $x_1, \dots, x_n$. Denote by $u(t, x_1, \dots, x_n, \lambda, t_0; f)$ (more briefly $u(t, x, \lambda, t_0; f)$) the solution of the differential-operator equation*

$$(T - \lambda X)u = 0 \tag{1}$$

(λ is a parameter), satisfying the initial conditions

$$D^q u\big|_{t=t_0} = \begin{cases} 0, & q = 0, \dots, p-2, \\ f(x), & q = p-1, \end{cases} \qquad x = (x_1, \dots, x_n). \tag{2}$$

Consider the factorized differential-operator equation

$$(T - \lambda_1 X)^{r_1} \dots (T - \lambda_{lX})^{r_l} w = 0. \tag{3}$$

Here $\lambda_1, \dots, \lambda_l$ ($l \geqslant 1$) are pairwise distinct parameters; $r_1, \dots, r_l$ are positive integers—the degrees of the corresponding operators, $r_1 + \dots + r_l = m > 1$.

Let $w(t, x, \lambda_1, \dots, \lambda_l, t_0; f)$ be the solution of equation (3) satisfying the conditions

$$D^q w\big|_{t=t_0} = \begin{cases} 0, & q = 0, \dots, mp-2, \\ f(x), & q = mp-1. \end{cases} \tag{4}$$

Below an explicit representation of $w(t, x, \lambda_1, \dots, \lambda_l, t_0; f)$ in terms of $u(t, x, \lambda_1, t_0; f), \dots, u(t, x, \lambda_l, t_0; f)$ is established, and some of its applications are indicated.

2. Let $K_1(t,\tau)$ be the solution of the ordinary differential equation

$$T_t K_1(t,\tau) = 0, \tag{5}$$

satisfying the conditions

$$D^q K_1\big|_{t=\tau} = \begin{cases} 0, & q = 0, \dots, p-2, \\ 1, & q = p-1; \end{cases} \tag{6}$$

and

$$K_s(t,\tau) = \int_\tau^t K_{s-1}(t,z) K_1(z,\tau)\, dz, \qquad s = 2, \dots, m-1. \tag{7}$$

There holds the formula

$$w(t,x,\lambda_1,\dots,\lambda_l,t_0;f) =$$
$$= \int_{t_0}^t K_{m-1}(t,\tau) \sum_{k=1}^{l} \frac{1}{r_k - 1!} \frac{\partial^{r_k-1}}{\partial \lambda_k^{r_k-1}} \left(\frac{\lambda_k^{m-1}}{\prod_{i=1}^{l}(\lambda_k - \lambda_i)^{r_i}} u(\tau,x,\lambda_k,t_0;f) \right) d\tau. \tag{8}$$

* We shall omit the subscript t on the operator T where this cannot cause misunderstanding.

under the assumption that: 1) the functions $a_k(t) \in C_{p(m-1)-1}$ $(k = 1,\dots,p)$ on the interval G_t under consideration in the variable t, $t_0 \in G_t$; 2) the operator T in G_t, the operator X and the function $f(x)$ in the domain G_x under consideration in the variable x are such as to ensure the smoothness in λ, required on the right-hand side of (8), of the function $u(t,x,\lambda,t_0;f)$ for $\lambda = \lambda_1,\dots,\lambda_l$; 3) the operator X $(\neq 0)$ is permutable with integration with respect to τ and with differentiations with respect to λ_k on the right-hand side of (8); 4) $D^k D_j^s u_j = D_j^s D^k u_j$, $D_j = \partial/\partial\lambda_j$, $u_j = u(t,x,\lambda_j,t_0;f)$, $0 < k \le p$, $0 < s \le r_j - 1$ $(j = 1,\dots,l)$. (A prime on a product means omission of the factor corresponding to the value $i = k$.)

In particular, for $l = 1$, i.e., for the Cauchy problem (4) referring to the iterated equation

$$(T - \lambda X)^m w = 0, \tag{3'}$$

we obtain the formula*

$$w(t,x,\lambda,t_0;f) = \int_{t_0}^t K_{m-1}(t,\tau) \frac{\partial^{m-1}}{\partial\lambda^{m-1}} \left(\frac{\lambda^{m-1}}{(m-1)!} u(\tau,x,\lambda,t_0;f) \right) d\tau. \tag{8'}$$

Remark 1. If X in equation (3) is an operator of multiplication ($X \neq 0$), the representation of w in terms of u_j ($j = 1, \dots, l$) can, under conditions 1), 2), and 4), be written in the form

$$w(t, x, \lambda_1, \dots, \lambda_l, t_0; f) = \sum_{k=1}^{l} \frac{1}{r_k - 1!} \frac{\partial^{r_k - 1}}{\partial \lambda_k^{r_k - 1}} \left(\frac{X^{-(m-1)} u(t, x, \lambda_k, t_0; f)}{\prod_{i=1}^{l}{}' (\lambda_k - \lambda_i)^{r_i}} \right). \quad (9)$$

Remark 2. Formula (8) (as also (9)) remains valid if the right-hand sides of equations (1) and (3) are replaced by one and the same function $F(t, x)$, and u and w are then understood as solutions of the corresponding nonhomogeneous equations satisfying respectively the conditions (2) and (4).

3. The problem of determining the solution $w(t, x, \lambda_1, \dots, \lambda_l; f_0, \dots, f_{mp-1})$ of equation (3), satisfying the general initial conditions

$$D^q w\big|_{t=t_0} = f_q(x), \qquad q = 0, \dots, mp - 1,$$

can be reduced (see (1)) to problem (3)–(4).

4.1. To establish the validity of formula (8), we first note the following lemmas.

Lemma 1. If the function $K_1(t, \tau)$ satisfies equation (5) and conditions (6), then, as a function of τ, $\tau \in G_t$, it satisfies (under the assumption that in G_t $a_k(t) \in C_{p-k}$ ($k = 1, \dots, p$)) the equation (see (2))

$$T_\tau^* K_1(t, \tau) = 0 \quad (10)$$

and the conditions

$$\widetilde{D}^q K_1\big|_{\tau=t} = (-1)^{p-1} \delta_{q,p-1}, \qquad q = 0, \dots, p-1, \quad \widetilde{D} = \partial/\partial\tau, \quad (11)$$

where T^* is the operator adjoint to the operator T, and $\delta_{q,p-1}$ is the Kronecker symbol.

Lemma 2. For $K_\alpha(t, \tau)$, ($\alpha = 2, \dots, m-1$), the equalities

$$T_t K_\alpha(t, \tau) = K_{\alpha-1}(t, \tau), \qquad D^q K_\alpha\big|_{t=\tau} = 0, \qquad q = 0, \dots, p-1, \quad (12)$$

$$T_\tau^* K_\alpha(t, \tau) = K_{\alpha-1}(t, \tau), \qquad \widetilde{D}^q K_\alpha\big|_{\tau=t} = 0, \qquad q = 0, \dots, p-1. \quad (13)$$

hold. Hence, in particular, it follows that $K_{m-1}(t, \tau)$ satisfies the equation

$$T_t^{m-1} K_{m-1}(t, \tau) = 0. \quad (14)$$

* The case when in equation (3) $l = 1$ was considered in (1); the integral representation of w in terms of u obtained there contains no differentiations with respect to λ, but the kernel in it is defined as the solution of the corresponding equation in partial derivatives.

and the conditions

$$D^q K_{m-1}\big|_{t=\tau} = \delta_q, \quad p(m-1)-1,\ q = 0, \dots, p(m-1)-1. \tag{15}$$

Lemma 3. Let

$$a_{kl}^{(m)} = \frac{\lambda_k^{m-1}}{r_k - 1!} \prod_{i=1}^{l}{}' (\lambda_k - \lambda_i)^{-r_i}; \tag{16}$$

$\lambda_1, \dots, \lambda_l$ are all distinct from one another; r_i $(i = 1, \dots, l)$ are integers ≥ 1; $r_1 + \dots + r_l = m$.

The identities hold

$$\sum_{k=1}^{l} \frac{\partial^{r_k - 1}}{\partial \lambda_k^{r_k - 1}} \left(\frac{a_{kl}^{(m)}}{\lambda_k^s} \right) = \begin{cases} 0, & s = 1, \dots, m-1, \\ 1, & s = 0. \end{cases} \tag{17}$$

4.2. We now outline the proof of formula (8). Relying on the assumptions of § 2, (1)–(2) for $\lambda = \lambda_1, \lambda_2$, (5), (6), (10), and (11), we establish that the function w, defined by formula (8) for $m = 2$ and $r_1 = r_2 = 1$ (which we denote by w_2), satisfies the equation $(T - \lambda_2 X) w_2 = u(t, x, \lambda_1, t_0; f)$, and consequently also the corresponding equation (3). Next, let in equation (3) $l = \sigma$ and $r_k = 1$ $(k = 1, \dots, \sigma)$, $\sigma > 2$; denote the corresponding function w (according to formula (8)) by w_σ, and suppose that it satisfies the corresponding equation (3). Then, using (1)–(2) for $\lambda = \lambda_1, \dots, \lambda_{\sigma+1}$, (12), (13), and (17), under the assumptions of § 2 we find that $(T - \lambda_{\sigma+1} X) w_{\sigma+1} = w_\sigma$, i.e. $w_{\sigma+1}$ satisfies equation (3) for $l = \sigma + 1$, $r_k = 1$, $k = 1, \dots, \sigma + 1$. Taking into account what has been proved for the function w_2, we obtain that w_l for $r_1 = \dots = r_l = 1$ satisfies the corresponding equation (3). Further we establish, as above, that if the function $w = w_{\rho_1, \dots, \rho_l}$ (defined by formula (8)) is a solution of equation (3) with the exponents $r_1, \dots, r_l$ in it respectively equal to $\rho_1, \dots, \rho_l$, then the function $w_{\rho_1 + 1, \rho_2, \dots, \rho_l}$ satisfies the equation

$$(T - \lambda_1 X) w_{\rho_1 + 1, \rho_2, \dots, \rho_l} = w_{\rho_1, \rho_2, \dots, \rho_l},$$

and consequently also the corresponding equation (3). In view of the commutativity of the operators $(T - \lambda_i X)^{r_i}$, $(T - \lambda_j X)^{r_j}$, it is possible to restrict the induction to only one of the exponents. Since $w_{1, \dots, 1}$ satisfies the corresponding equation (3), this completes the proof that the function w, defined by formula

(8), satisfies equation (3). As for the conditions (4), their fulfillment (under the assumptions of § 2) can be verified for: a) $q = 0, \dots, p(m-1)-1$; b) $q = p(m-1)+i$; $i = 0, \dots, p-2$, and c) $q = pm-1$, if one takes into account respectively (15), (2) and, finally, (2) and (17).

Let us note some applications of formula (8).

5. Let in equation (3) the operator $T = D^p$; in this case

$$K_{m-1}(t,\tau) = \frac{1}{s!}(t-\tau)^s, \qquad s = p(m-1)-1. \tag{18}$$

5.1. If $l = m$, i.e. $r_1 = \dots = r_m = 1$, then (see (8) and (16))

$$w = \int_0^t \frac{(t-\tau)^s}{s!} \sum_{k=1}^m a_{km}^{(m)} u(\tau, x, \lambda_k, 0; f)\, d\tau \tag{19}$$

(here and below in §§ 5–6 it is assumed that $t_0 = 0$). In particular, for $p = 1$ and $\lambda_k = e^{2\pi ik/m}$ $(k = 1, \dots, m)$ we have $a_{km}^{(m)} = 1/m$, and (19) gives, for $X = \partial/\partial x$, a generalization of d' Alembert' s formula (under the assumption that for $m > 2$ the function $f(x)$ is analytic) (3). For $p = 1$, $X = \Delta$, $m = 2$, and $\lambda_{1,2} = \pm i$, we obtain (under the appropriate smoothness assumptions on $f(x)$) a solution of the generalized Boussinesq problem (4).

5.2. Let in equation (3) at least one of the numbers $r_1, \dots, r_l$ be greater than 1; the differentiations of the function $u(\tau, x, \lambda_k, 0; f)$ that arise in this case

with respect to λ_k in formula (8) can be (for $T = D^p$) reduced to such differentiations with respect to τ:

$$\frac{\partial^j u(t, x, \lambda_k, 0; f)}{\partial \lambda_k^j} = (\lambda_k p)^{-j} t^p \left(\frac{\partial}{\partial t} t^{1+p}\right)^j \left(t^{-(j+1)p} u(t, x, \lambda_k, 0; f)\right), \tag{20}$$

as a result of which (taking into account (15), (2), and (16)) we can give formula (8) the form

$$w = \int_0^t \sum_{k=1}^l \psi_k(t, \tau, \lambda_1, \dots, \lambda_l) u(\tau, x, \lambda_k, 0; f)\, d\tau \quad (p > 1 \text{ when } l = 1), \tag{21}$$

where

$$\psi_k(t, \tau, \lambda_1, \dots, \lambda_l) =$$

$$= \sum_{j=0}^{r_k-1} (-\lambda_k p)^{-j} C_{r_k-1}^{j} \frac{\partial^{r_k-1-j} a_{kl}^{(m)}}{\partial \lambda_k^{r_k-1-j}} \tau^{-(j+1)p} \left(\tau^{1+p} \frac{\partial}{\partial \tau}\right)^j (\tau^p K_{m-1}(t,\tau)). \quad (22)$$

For $p = 1$ and $l = 1$ (i.e. for the Cauchy problem (4) relating to equation (3′) with $C = D$), from (8′), taking (20) into account, we obtain $w = \frac{t^{m-1}}{m-1!} \times u(t, x, \lambda, 0; f)$; for the case when X is the Laplace operator, see (5).

For $p > 1,\ l = 1$, it follows from (22) that

$$\psi_1(t,\tau) = \frac{(-1)^{m-1}}{p^{m-1}(m-1)!} \tau^{-p} \left(\tau^{p+1} \frac{\partial}{\partial \tau}\right)^{m-1} \left(\tau^{-p(m-2)} K_{m-1}(t,\tau)\right); \quad (23)$$

in particular, for $p = 2$,

$$\psi_1(t,\tau) = 2^{3-2m}[(m-1)!(m-2)!]^{-1} \tau (t^2 - \tau^2)^{m-2},$$

and (21) gives the structure of the solution of the Cauchy problem (4) for the "generalized polywave" equation (3′) (see (1)). The case $p = 2,\ l \geq 1$, under the assumption that X in equation (3) is the Laplace operator, was considered in (6); there w only for $r_1 = \cdots = r_m = 1$ is directly expressed in terms of $u(t, x, \lambda_1, 0; f), \ldots, u(t, x, \lambda_m, 0; f)$.

Representation (21) makes it possible to establish all cases when, for the solution of problem (3)–(4) with

$$T = D^2, \qquad X = \sum_{i=1}^{n} \left(\partial^2/\partial x_i^2 + \frac{a_i}{x_i} \partial/\partial x_i\right),$$

$\lambda_k > 0,\ k = 1, \ldots, l$ (a_i are constants and, for $a_i \neq 0$, $|x_i| > t \geq 0$ $(i = 1, \ldots, n)$), the Huygens principle holds (cf. (7); (1) for $l = 1$, and (6) for $a_i = 0$ $(i = 1, \ldots, n),\ l \geq 1$).

6. For the case when the operator in the left-hand side of (3) consists of "generalized wave factors with dispersion," i.e. $T = D^2 + \mu^2$, μ is constant, we have

$$K_{m-1}(t,\tau) = c(t-\tau)^s J_s(\mu(t-\tau)), \qquad 2s = 2m - 3,$$

$$c = (\sqrt{\pi}/2\mu)^s/(m-2)!.$$

In particular, if the operator X is the Laplacian, formula (8′) gives the solution of the Cauchy problem (4) for the iterated Klein-Gordon equation (cf. (8)).

7. Let us note that, for the case when

$$T = D^p + \frac{a_1}{t} D^{p-1} + \dots + \frac{a_{p-1}}{t^{p-1}} D + a_p,$$

$a_1, \dots, a_p$ are constants and $t_0 > 0$ (i.e. when (1)–(2) is a regular Cauchy problem for the generalized Euler-Poisson-Darboux differential-operator equation (9)), the differentiations of $u(t, x, \lambda_k, t_0, f)$ with respect to λ_k in (8) reduce to such differentiations with respect to t and t_0:

$$p\lambda_k \partial u / \partial \lambda_k = (t\partial/\partial t + t_0 \partial/\partial t_0 + 1 - p)\, u.$$

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Note: Figure translations are in progress. See original paper for figures.