HARMONIC ANALYSIS ON THE ONE-SHEETED HYPERBOLOID

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Abstract

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MATHEMATICS

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HARMONIC ANALYSIS ON THE ONE-SHEETED HYPERBOLOID

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1. Let a bilinear form be given in the n-dimensional real space R^n: $[x,y] = -x_1y_1 + x_2y_2 + \cdots + x_{ny}n$; let G be the connected group of linear transformations with determinant 1 that preserve this form. G acts transitively on the one-sheeted hyperboloid $X : [x,x] = 1$. The shifts $f(x) \to f(xg)$ form a unitary representation of G in $L^2(X)$ with respect to the invariant measure $dx = |x_1|^{-1} dx_2 \cdots dx_n$. X is an example of a homogeneous space with a noncompact stationary subgroup. Harmonic analysis on such spaces has been studied only in special cases. An analogue of the Plancherel formula on the one-sheeted hyperboloid for $n = 4$ (G is the Lorentz group) was obtained by the horisphere method in ([1]) (see also ([2])).

In the present work $L^2(X)$ is decomposed into irreducibles with the aid of spherical functions.

Let W be the stationary subgroup of the point $x^0 = (0, 1, 0, \dots, 0)$; let T_g be an irreducible unitary representation of G in a Hilbert space H, and let θ be a linear functional defined on an everywhere dense set D and invariant with respect to W. Then the expression $(T_g\theta, \theta)$, understood in the regularized sense (see § 5), defines a function on X: $\Phi(x) = (T_g\theta, \theta)$, $x = x^0 g$, which is called spherical. The main problem consists in expanding the δ-function on X in spherical functions.

2. **Representations of G associated with the cone.** Representations of the group G were studied in ([3–5]) (see also ([6]), Ch. X, § 2).

Let τ be a complex number; D_τ the space of infinitely differentiable functions $\varphi(\xi)$ on the upper part of the cone $X_0^+ : [x,x] = 0$, $x_1 > 0$, homogeneous of degree $\sigma = (2-n+\tau)/2$: $\varphi(t\xi) = t^\sigma \varphi(\xi)$, $t > 0$. In D_τ there acts a representation T^τ of the group G: $\varphi(\xi) \to \varphi(\xi g)$. A function $\varphi \in D_\tau$ is determined by its values on the set $Y : \xi_1 - \xi_2 = 1$; a point $\eta \in Y$ has the form

$$\eta = (\eta_1, \eta_2, \eta) = \left(\frac{1}{2}(\|\eta\|^2 + 1), \frac{1}{2}(\|\eta\|^2 - 1), \eta_3, \dots, \eta_n \right),$$

where $\|\eta\|^2 = \eta_3^2 + \cdots + \eta_n^2$. D_τ is realized as a certain aggregate of functions from $C^\infty(Y)$.

The operator

$$A_\tau : \varphi(\eta) \to \frac{1}{\Gamma(-\tau/2)} \int \|\eta - \eta'\|^{2-n-\tau} \varphi(\eta')\, d\eta' \qquad (d\eta = d\eta_3 \cdots d\eta_n)$$

maps D_τ into $D_{-\tau}$ and commutes with the action of G.

Continuous series. $\tau = i\rho$, $-\infty < \rho < \infty$. T^τ is irreducible and preserves the scalar product

$$(\varphi, \varphi) = \int |\varphi(\eta)|^2\, d\eta,$$

T^τ and $T^{-\tau}$ are equivalent.

Discrete series. For $\tau = 2 - n - 2l$, $l = 0, 1, 2, ...$, the operator A_τ has in D_τ its own null subspace F_τ, which is irreducible with respect to the restriction of T^τ (except for $n = 3$) and in which there exists an invariant scalar product

$$B(\varphi, \varphi) = c_{nl} \int \|\eta - \eta'\|^{2l} \nu_n \|\eta - \eta'\| \varphi(\eta) \overline{\varphi(\eta')}\, d\eta\, d\eta'.$$

The constant c_{nl} here is chosen equal to $(-1)^{l+1} 2\pi^{2-n} \Gamma(l{+}n{-}2)/\Gamma(l{+}1)\Gamma(-\tau/2)$. The representations in F'_τ form a discrete series.

D_{ip} and F_{2-n-2l} are completed to Hilbert spaces H_p and H_l with respect to the scalar products already present.

3. Functions invariant with respect to the stationary subgroup. The representation T^τ is extended from D_τ to the space D'_τ of generalized functions $\theta(\eta)$ on X_0^+, homogeneous of degree σ, $\theta : \varphi(\eta) \to \int \theta(\eta) A_\tau \varphi(\eta)\, d\eta$. For $\tau = 2 - n - 2l$ denote by F'_τ the null subspace of the operator A_τ. In D'_τ there exist two linearly independent functions invariant with respect to W, namely $(\xi_2)^\sigma_\pm$, if $\sigma \neq -m$, m natural, and ξ_2^{-m}, $\delta^{(m-1)}(\xi_2)$, if $\sigma = -m$. Only one of the functions ξ_2^{2-n-l}, $\delta^{(n-3+l)}(\xi_2)$ belongs to F'_τ: the first for odd n, the second for even n. Put $\theta_l(\eta) = \beta_l(\eta_2)$, where $\beta(t) = (-1)^{(n-3)/2} t^{2-n-l}$ for odd n, $\beta(t) = (-1)^t l \Gamma^{-1}(l + n - 2) \delta^{(n-3+l)}(t)$ for even n.

4. Integral representation of spherical functions. Let

$$V = \{v \in R^n : [v, v] = 0,\ v_2 = 1\}, \qquad V^\pm = V \cap \{v_1 \gtrless 0\}, \qquad \tilde\eta = \eta_2 v.$$

Continuous series. To the generalized functions $\theta^\pm_p(\eta) = (\eta_2)^\sigma_\pm$ there correspond (see §1) the spherical functions

$$\Phi^\pm_\rho(x) = \int_Y [x^0, \tilde\eta]^\sigma_\pm [x, \tilde\eta]^{\bar\sigma}_\pm\, d\eta = \int_{V^\pm} [x, v]^{\bar\sigma}_+\, dv \qquad \left(dv = \frac{dv_3 \dots dv_n}{|v_1|}\right). \quad (1)$$

Discrete series. From $\theta_l(\eta)$ we obtain (see §1) the spherical functions

$$\Psi_l(x) = p_{nl} \int_Y [x_0, \tilde\eta]^l_+\, \beta_l([x, \tilde\eta])\, d\eta = p_{nl} \int_{V^+} \beta_l([x, v])\, dv, \qquad (2)$$

where $p_{nl} = 2^{-\tau}\pi^{(4-n)/2}/\Gamma(l+1)$. (For $n = 3$, Ψ_l is the sum of two spherical functions.) The integrals (1), (2) are understood in the sense of the regularized value, see §5.

5. Regularization of integrals. Consider the integral

$$R_\lambda(x) = \int_{V^+} [x, v]_+^\lambda \, dv,$$

where λ is complex, $x \in X$. If $|x_2| > 1$, then it converges absolutely for values of λ lying in a certain domain, and is analytically continued to the points σ of interest to us from (1). If $|x_2| < 1$, then $R_\lambda(x)$ diverges for all λ. However, then the integral

$$R_{\lambda,\mu}(x) = \int_{V^+} [x, v]_+^\lambda |v_1|^\mu \, dv$$

converges absolutely for $\mathrm{Re}(\lambda + \mu) < 3 - n$, $\mathrm{Re}\,\lambda > -1$, and is analytically continued to the point $\lambda = \sigma$, $\mu = 0$. We proceed analogously with the integrals (2) and (3).

6. Expression of the spherical functions in terms of Legendre functions.

Continuous series. Let $\nu = (n-3)/2$, $E(\rho) = \dfrac{2}{\pi}(2\pi)^\nu |\Gamma(\sigma+1)|^2$. Then

$$\frac{\Phi_\rho^+(x)}{E(\rho)} = \begin{cases} -\,e^{-i\nu\pi} \sin\sigma\pi\,(x_2^2-1)^{-\nu/2} Q_{\nu+\sigma}^\nu(x_2), & \text{for } x_2 > 1,\ x_1 > 0, \\ -\,e^{-i\pi\nu} \sin\bar{\sigma}\pi\,(x_2^2-1)^{-\nu/2} Q_{-\nu-\bar{\sigma}-1}^\nu(x_2), & \text{for } x_2 > 1,\ x_1 < 0, \\ \dfrac{\pi}{2}(1-x_2^2)^{-\nu/2} P_{\nu+\sigma}^\nu(-x_2), & \text{for } |x_2| < 1, \\ 0, & \text{for } x_2 < -1,\ x_1 > 0, \\ \pi\cos\pi\nu\,(x_2^2-1)^{-\nu/2} P_{\nu+\sigma}^\nu(-x_2), & \text{for } x_2 < -1,\ x_1 < 0. \end{cases}$$

The expressions for $\Phi_\rho^-(x)$ are obtained by replacing x_1 by $-x_1$.

Discrete series. If we denote

$$f(z) = (z^2-1)^{-\nu/2} Q_{l+\nu}^\nu(z),$$

then

$$\Psi_l(x) = 2^{\nu-l}\overline{\gamma_l}\Gamma^{-1}(l+n-2)\,[f(x_2+i0) - (-1)^n f(x_2-i0)].$$

All spherical functions are locally integrable.

7. Expansion of the δ-function in spherical functions. Let $\delta(x)$ denote the δ-function on X, concentrated at x^0. For the expansion of $\delta(x)$ in spherical functions we use the method of M. Riesz, as in ([1],[2]). Take the generalized function of one variable a

$$s(a,\lambda) = \frac{1}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}\,|a-1|^{2\lambda} a_+^{-\lambda-\nu}, \qquad \nu = (n-3)/2,$$

and put

$$S(x,\lambda)=\int_Y s([x,v],\lambda)\,dv,\qquad x\in X. \tag{3}$$

The integral is understood in the sense of the regularization of Sect. 5. It is computed explicitly:

$$S(x,\lambda)=\{\cos\pi\lambda\cdot(1-x_2)_-^{\lambda}+(1-x_2)_+^{\lambda}\}S_1(x)S_2(\lambda), \tag{4}$$

where $S_1(x)=(x_2+1)^{-\nu}$ for odd n; $S_1(x)=(x_2+1)_+^{-\nu}$ for even n;

$$S_2(\lambda)=\pi^{-1/2-\nu}2^{-\lambda+\nu+1}\Gamma(-\nu-\lambda).$$

Let $K(\lambda,\rho)$ be such that

$$s(a,\lambda)=\int_{-\infty}^{\infty}a_+^{(2-n+i\rho)/2}K(\lambda,\rho)\,d\rho.$$

Then, denoting $\Phi_\rho=\Phi_\rho^+ +\Phi_\rho^-$, we obtain from (1), (2)

$$S(x,\lambda)=\int_{-\infty}^{\infty}K(\lambda,\rho)\overline{\Phi_\rho(x)}\,d\rho. \tag{5}$$

Put $\lambda=-\nu-1$. Using the results of (7), Ch. III, § 2, item 2, and also the relation $\Phi_\rho=\Phi_{-\rho}$, we obtain from (4) and (5)

$$\varepsilon_n\delta(x)=\tfrac{1}{2}\pi^{-\nu-2}\Gamma(\nu+1)L(x)+\int_{-\infty}^{\infty}\omega(\rho)\overline{\Phi_\rho(x)}\,d\rho,$$

where

$L(x)=(x_2-1)^{-1}(1-x_2^2)^{-\nu}$ for odd n; $\quad L(x)=(x_2-1)^{-1}(1-x_2^2)_+^{-\nu}$ for even n;

$\varepsilon_n=\frac{1}{2}$ or 1, respectively, and

$$2^{2n-3}\pi^{n-1}\omega(\rho)=\begin{cases}\rho\tanh\dfrac{\pi\rho}{2}\cdot\displaystyle\prod_{k=1}^{\nu}[(2k-1)^2+\rho^2], & n\text{ odd},\\ \displaystyle\prod_{k=0}^{[\nu]}(4k^2+\rho^2), & n\text{ even}.\end{cases}$$

We expand $L(x)$ in $\Psi_l(x)$, for which purpose we use the expressions of Ψ_l in terms of Legendre functions. Finally:

$$\delta(x)=\sum_{l=0}^{\infty}a_l\overline{\Psi_l(x)}+\int_{-\infty}^{\infty}\omega(\rho)\overline{\Phi_\rho(x)}\,d\rho,$$

where

$$a_l=\tfrac{1}{2}\pi^{-n/2-1}(2l+n-2)\Gamma(l+n-2)\,4^{2-n-l}.$$

8. An analogue of the Plancherel formula. Let $C_c^\infty(X)$ be the space of compactly supported infinitely differentiable functions on X. We shall call the following functions on Y the Fourier components of a function $f \in C_c^\infty(X)$:

$$F_\rho^\pm(\eta) = \int f(x)[x,\eta]_\pm^\sigma\,dx, \qquad \sigma = \frac{2-n+i\rho}{2},$$

$$F_l(\eta) = \int f(x)\beta_l([x,\eta])\,dx.$$

For each ρ or l, these formulas define a mapping of $C_c^\infty(X)$ into D_τ, $\tau = i\rho$ or $\tau = 2-n-2l$, respectively. Under this, the shifts by g pass into T_g^τ.

From the expansion of $\delta(x)$ and the integral representations (1), (2), it follows that $f \in C_c^\infty(X)$ is reconstructed from its Fourier components as follows:

$$f(x) = \sum_{l=0}^{\infty} a_l B(F_l, \beta_l, ([x,\tilde\eta])) + \int_{-\infty}^{\infty} \omega(\rho)\{(F_\rho^+, [x,\tilde\eta]_+^\sigma) + (F_\rho^-[x,\tilde\eta]_-^\sigma)\}\,d\rho.$$

The correspondence

$$f \to \{F_\rho^+,\, F_\rho^-,\, F_l\}$$

can be extended to

$$L^2(X) \to \{H_\rho,\, H_\rho,\, H_l\}.$$

Thus, the representation in $L^2(X)$ of the group G decomposes into representations of the continuous series with multiplicity 2 and representations of the discrete series with multiplicity 1. There is an analogue of the Plancherel formula

$$\int |f(x)|^2\,dx = \sum_{l=0}^{\infty} a_l B(F_l, F_l) + \int_{-\infty}^{\infty} \omega(\rho)\{(F_\rho^+, F_\rho^+) + (F_\rho^-, F_\rho^-)\}\,d\rho,$$

where a_l, ω_ρ are determined by the formulas preceding (7).

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Note: Figure translations are in progress. See original paper for figures.