ON THE ADDITIVE THEORY OF IDEALS

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Abstract

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ON THE ADDITIVE THEORY OF IDEALS

IN RINGS, MODULES, AND GROUPOIDS

In the present note the concept of a system with quotients is introduced, which is a generalization of the (σ)-algebras of Lesieur and Croisot ([1]). As is known, the concept of a (σ)-algebra made it possible simultaneously to develop an additive theory for ideals of an associative ring, ideals of a semigroup, and submodules over an associative ring. With the aid of a system with quotients, the additive theory is developed simultaneously for ideals of a not necessarily associative ring, ideals of a groupoid, submodules of a module over a not necessarily associative ring, and normal divisors of a group (taking commutator multiplication ([2])). In systems with quotients the concept of s-primeness is introduced, encompassing most of the known generalizations of classical primeness. It is proved that, under certain natural requirements, in our opinion, the concept of tertiariness introduced by Lesieur and Croisot is the unique "good" generalization of classical primeness (cf. ([3])).

Let R and L be two arbitrary structures (lattices). We shall say that a system (R, r, l, L) with quotients is given to us if: 1) to every two elements $a \in R$, $\alpha \in L$ there is assigned a (uniquely determined) element $r(a, \alpha) \in R$, called the right quotient of a by α; 2) to every two elements $a, b \in R$ there is assigned a (uniquely determined) element $l(a, b) \in L$, called the left quotient of a by b; 3) for all $a, b, c \in R$, $\alpha, \beta \in L$,

$$b \leqslant r(a, l(a, b)), \qquad \alpha \leqslant l(a, r(a, \alpha)), \qquad a \leqslant r(a, \alpha), \qquad \alpha \leqslant l(a, a),$$

$$r(a, \alpha) \wedge r(b, \alpha) = r(a \wedge b, \alpha), \qquad r(a, \alpha) \wedge r(a, \beta) = r(a, \alpha \vee \beta),$$

$$l(a, c) \wedge l(b, c) = l(a \wedge b, c), \qquad l(a, b) \wedge l(a, c) = l(a, b \vee c).$$

In what follows an arbitrary system with quotients satisfying the condition below is considered:

(μ) The structure R is Dedekind, and moreover every element $a \in R$ can be represented as the intersection of a finite number of irreducible* elements $a_i \in R$; for any element $a \in R$ the maximality (and hence also minimality) condition is satisfied for right quotients $r(a, \alpha)$ and left quotients $l(a, b)$.

A system (R, r, l, L) with quotients will satisfy condition (μ), for example, in the following two important cases:

a) the structure R satisfies the maximality and Dedekind conditions, and the structure L satisfies the maximality condition for the left quotients of any element $a \in R$;

b) the structure R is complete, Dedekind, satisfies the minimality condition, and is continuous with respect to intersections; the structure L satisfies the minimality condition for the left quotients of any element $a \in R$.

Let (R, r, l, L) be an arbitrary system with quotients satisfying condition (μ). An element $a \in R$ will be called **prime** if it cannot–

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* An element $a \in R$ is called **irreducible** if it cannot be represented as the intersection of two strictly larger elements $b, c \in R$.

can be represented as the intersection of two strictly larger right quotients $r(a, \alpha)$, $r(a, \beta)$. A left quotient $l(a, b)$ is called **proper** if b is not contained in a. By condition (μ), every proper left quotient $l(a, b)$ is contained in some maximal proper left quotient $l(a, c)$. For any element $a \in R$, $a \neq 1$*, put

$$pl(a) = \bigwedge \{\alpha \mid \alpha \in \mathfrak{M}_a\},$$

where $\mathfrak{M}_a$ is the set of all maximal proper left quotients $l(a, c)$ of the element a. In addition, put $pl(1) = \varepsilon$.

Proposition 1. For every element $a \in R$ the element $pl(a)$ is the greatest among the elements $\alpha \in L$ such that, for every $\beta \in L$, from

$$r(a, \alpha) \wedge r(a, \beta) = a$$

it follows that $r(a, \beta) = a$. For every element $a \in R$, $a \neq 1$, the following assertions are equivalent:

a) the element a is a primal element;

b) the set $\mathfrak{M}_a$ consists of one element, i.e. there exists a greatest proper left quotient $l(a, c)$ of the element a (in this case, obviously, $\mathfrak{M}_a = \{pl(a)\}$);

c) for every $\alpha \in L$ the inclusions $r(a, \alpha) > a$, $\alpha \leq pl(a)$ are equivalent.

A representation $a = a_1 \wedge \cdots \wedge a_n$ of an element $a \in R$ as the intersection of a finite number of primal elements a_i will be called **primal-reduced** if no a_i can be omitted (i.e. for every a_i,

$$a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_n > a),$$

and no a_i can be replaced by a strictly larger right quotient $r(a_i, \alpha_i)$ (i.e. if $r(a_i, \alpha_i) > a_i$, then

$$a_1 \wedge \cdots \wedge a_{i-1} \wedge r(a_i, \alpha_i) \wedge a_{i+1} \wedge \cdots \wedge a_n > a),$$

and moreover the elements $\rho_i = pl(a_i)$ are pairwise incomparable.

Theorem 1. For every element $a \in R,\ a \neq 1$, there exists at least one primal-reduced representation. If

$$a = a_1 \wedge \cdots \wedge a_n$$

and

$$a = a'_1 \wedge \cdots \wedge a'_m$$

are two primal-reduced representations of the element $a \in R,\ a \neq 1$, then $m = n$, and the elements a'_i can be renumbered so that for all i

$$\rho_i = pl(a_i) = pl(a'_i).$$

In order that an element $\rho \in L$ coincide with some ρ_i, it is necessary and sufficient that ρ be a maximal proper left quotient $l(a, c)$ of the element a.

An element $a \in R$ is called **tertiary** if it cannot be represented as the intersection of a strictly larger right quotient $r(a, \alpha)$ and a strictly larger element $b \in R$. A left proper quotient ρ of an element $a \in R$ is called **essential** if there exists an element $b \in R,\ b > a$, such that $\rho = l(a, b)$, and for every $c \in R$ from $b \geq c > a$ it follows that

$$\rho = l(a, b) = l(a, c).$$

For every element $a \in R,\ a \neq 1$, put

$$ter(a) = \bigwedge \{\alpha \mid \alpha \in \mathfrak{Z}_a\},$$

where $\mathfrak{Z}_a$ is the set of all essential left quotients $l(a, c)$ of the element a. In addition, put $ter(1) = \varepsilon$. Since every maximal proper left quotient

$$l(a, c) = l(a, a \vee c)$$

is essential, it follows that $pl(a) \geq ter(a)$.

Proposition 2. For every element $a \in R$ the element $ter(a)$ is the greatest among the elements $\alpha \in L$ such that, for every element $b \in R$, from

$$r(a, \alpha) \wedge b = a$$

it follows that $b = a$. For every element $a \in R,\ a \neq 1$, the following assertions are equivalent:

a) the element a is a tertiary element;

b) the set $\mathfrak{Z}_a$ consists of one element, i.e. there exists a unique essential left quotient $l(a, c)$ of the element a (in this case, obviously, $\mathfrak{Z}_a = \{ter(a)\}$);

c) for every $\alpha \in L$ the inclusions $r(a, \alpha) > a,\ \alpha \leq ter(a)$ are equivalent.

A representation $a = a_1 \wedge \cdots \wedge a_n$ of an element $a \in R$ as the intersection of a finite number of tertiary elements a_i will be called **tertiary-reduced** if no a_i can be omitted and the elements

$$\rho_i = ter(a_i) = pl(a_i)$$

are pairwise distinct.

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* If (R, r, l, L) is a system with quotients satisfying condition (μ), then, as is easy to show, the structure R has a greatest element 1, and the structure L has a greatest element ε.

Theorem 2. For any element $a \in R,\ a \neq 1$, there exists at least one tertiarily reduced representation. If $a = a_1 \wedge \cdots \wedge a_n$ and $a = a'_1 \wedge \cdots \wedge a'_m$ are two tertiarily reduced representations of $a \in R,\ a \neq 1$, then $m = n$, and the elements a'_j can be renumbered so that, for all i, $\mathrm{ter}(a_i) = \rho_i = \mathrm{ter}(a'_i)$. In order that an element $\rho \in L$ coincide with some ρ_i, it is necessary and sufficient that ρ be an essential left quotient of the element a.

Let us say that in the system (R, r, l, L) an s**-primarity** S is defined if with each $a \in R$ there is associated a uniquely determined element $s(a) \in L$ such that $s(1) = \varepsilon$ and $r(a, s(a)') > a$ for every $a \neq 1$. An element $a \in R,\ a \neq 1$, is called s**-primary** if, for every $\alpha \in L$, the inclusions $r(a, \alpha) > a$ and $\alpha \leq s(a)$ are equivalent. We shall regard the element 1 as s-primary for any s-primarity S. The set of s-primary elements for a given s-primarity S will be denoted by $I(S)$. Let us say that an s_1-primarity S_1 is equivalent to an s_2-primarity S_2 if $I(S_1) = I(S_2)$. We shall consider s-primarities up to equivalence. Putting $S_1 \geq S_2$ (the s_1-primarity S_1 is stronger than the s_2-primarity S_2) if and only if $I(S_1) \subseteq I(S_2)$, we introduce a partial ordering among s-primarities.

The notion of s-primarity generalizes the notions of tertiarity and primality. Namely, tertiarity is ter-primarity, and primality is pl-primarity.

Theorem 3. For any s-primarity S, an element $a \in R,\ a \neq 1$, is s-primary if and only if $s(a)$ is the greatest proper left quotient of the element a, and therefore for any s-primary element a, $s(a) = pl(a)$. The s-primarities S form a complete distributive lattice. Primality Pl is the weakest s-primarity. There exists a strongest s-primarity—the o-primarity O. In order that a set $I \subseteq R$ coincide with the set $I(S)$ for some s-primarity S, it is necessary and sufficient that $I(Pl) \supseteq I \supseteq I(O)$.

We consider the following conditions imposed on s-primarities:

C1. For any element $a \in R,\ a \neq 1$, there exists at least one s-representation, i.e. a representation in the form of an intersection of a finite number of s-primary elements a_i, in which no a_i can be omitted.

C2. For any element $a \in R,\ a \neq 1$, there exists at least one reduced s-representation—an s-representation $a = a_1 \wedge \cdots \wedge a_n$, such that the elements $s(a_i) = \rho_i$ are pairwise distinct.

C3. For any element $a \in R,\ a \neq 1$, there exists at least one partially extremal reduced s-representation—a reduced s-representation $a = a_1 \wedge \cdots \wedge a_n$, such that no a_i can be replaced by a strictly larger right quotient $r(a_i, \alpha_i)$.

C4. For any element $a \in R,\ a \neq 1$, there exists a strongly reduced s-representation—a partially extremal s-representation $a = a_1 \wedge \cdots \wedge a_n$, such that the elements ρ_i are pairwise incomparable.

E1. For any element $a \in R,\ a \neq 1$, any two reduced s-representations $a = a_1 \wedge \cdots \wedge a_n = a'_1 \wedge \cdots \wedge a'_m$ are equivalent: $m = n$, and the elements a'_j can be renumbered so that, for all i, $s(a_i) = s(a'_i)$.

E2. For any element $a \in R,\ a \neq 1$, any two partially extremal reduced s-representations are equivalent.

E3. For any element $a \in R,\ a \neq 1$, any two strongly reduced s-representations are equivalent.

P1. The intersection of a finite number of s-primary elements a_i with one and the same $s(a_i) = \rho$ is an s-primary element a with $s(a) = \rho$.

Theorem 4. In order that an s-primarity S be equivalent to primality Pl, it is necessary and sufficient that S satisfy condition C4. Every s-primarity S satisfies condition E3. In order that an s-primarity S be equivalent to tertiarity Ter, it is necessary and sufficient that it satisfy conditions C1, P1, and E1 (or C2, P1, and

E1). Tertiarity Ter is the strongest among the s-primarities S satisfying condition C3, the strongest among the s-primarities S satisfying conditions C1, P1, and the minimal among the s-primarities satisfying condition E2. The s-primarity S is equivalent to the tertiarity Ter if and only if S satisfies conditions C3, E2.

This theorem shows that tertiarity and primality are the only "good" s-primarities. Moreover, since the classical Noetherian primality (in associative-commutative Noetherian rings) satisfies conditions C1, P1 and E1, tertiarity is the only "good" generalization of classical primality.

Let us consider applications of the results obtained.

1. **Rings.** Let K be an arbitrary, not necessarily associative, ring; $L_k(D_k)$ the lattice of left (two-sided) ideals of the ring K. Quotients are defined as follows: if A, B, C are elements of the indicated lattices, then

$$r(A,B) = \bigcup\{C \mid BC \subseteq A\}, \qquad l(A,B) = \bigcup\{C \mid CB \subseteq A\}.$$

It is easy to verify that the systems (L_k, r, l, L_k) and (D_k, r, l, D_k) are systems with quotients. If requirement (μ) is fulfilled in these systems, then all the results proved above will hold. Similarly, one constructs the systems (Π_M, r, l, L_k) and (Π_M, r, l, D_k), where M is an arbitrary abelian group with a ring K of operators (a K-module), and Π_M is the lattice of admissible subgroups (submodules).

There exists a finite ring K such that: 1) in the system (L_k, r, l, L_k) the ideal 0 is a tertiary left ideal; 2) in the system (D_k, r, l, D_k) the ideal 0 is not a primal ideal (two-sided). There exists a finite ring K such that: 1) in the system (L_k, r, l, L_k) the ideal 0 is not a primal left ideal; 2) in the system (D_k, r, l, D_k) the ideal 0 is a tertiary ideal.

2. **Groupoids, semigroups.** Generally speaking, left (two-sided) ideals of a groupoid do not form a lattice if only nonempty ideals are considered. Therefore one considers either the lattice of all, including empty, left (two-sided) ideals, or only nonempty ideals are considered, but it is assumed that the groupoid has a zero. As for rings, we obtain the systems (L_G, r, l, L_G) and (D_G, r, l, D_G), where $G(\cdot)$ is an arbitrary groupoid satisfying the requirements stated above. In particular, semigroups may also be considered.

3. **Groups.** The approach indicated above is not of interest for groups. Therefore, for any group $G(\cdot)$ we consider the lattice N_G of normal divisors of the group G and introduce quotients as follows:

$$r(A, B) = l(A, B) = k(A, B) = \{x \in G \mid \text{for all } b \in B,\ [x, b] = xbx^{-1}b^{-1} \in A\}.$$

It is easy to verify that (N_G, k, k, N_G) is a system with quotients. We note only that in the lattice N_G union coincides with product. This system was considered in ([2]).

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Note: Figure translations are in progress. See original paper for figures.