DUALITY IN SPACES OF HOLOMORPHIC FUNCTIONS WITH SINGULARITIES

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Abstract

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MATHEMATICS

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DUALITY IN SPACES OF HOLOMORPHIC FUNCTIONS WITH SINGULARITIES

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1. Let O be a nonempty open subset of the complex plane, and let $\mathfrak{D}$ be the set of all closed discrete subsets of O. For each $D \in \mathfrak{D}$, let $H(O, D)$ be the vector space of all holomorphic functions on the set O with singularities in D, endowed with the topology of uniform convergence on every compact subset in $O \setminus D$. The set $\mathfrak{D}$ is filtered by inclusion $\subset$, and $H(O, D) \subset H(O, D')$ if $D \subset D'$, while the topology in $H(O, D)$ coincides with that induced from $H(O, D')$. The vector space

$$H(O, \mathfrak{D}) = \bigcup_{D \in \mathfrak{D}} H(O, D)$$

of all holomorphic functions on O with isolated singularities is endowed with the strongest of the locally convex topologies for which the canonical mappings

$$\varphi_D : H(O, D) \to H(O, \mathfrak{D}) \quad (D \in \mathfrak{D})$$

are continuous. The study of duality for the space $H(O, \mathfrak{D})$ is the subject of the present paper.

Let M be an arbitrary open subset in O, and let $\mathfrak{F}$ be the set of all finite subsets of M. The vector space $H(M, \mathfrak{F})$ of all holomorphic functions on M with a finite number of singularities, which is the union of the subspaces $H(M, F)$ $(F \in \mathfrak{F})$, is endowed with the strongest of the locally convex topologies for which the canonical mappings

$$\psi_F : H(M, F) \to H(M, \mathfrak{F}) \quad (F \in \mathfrak{F})$$

are continuous. Further, let $P(U)$, for every nonempty open subset U of the Riemann sphere S_2, denote the vector space of all holomorphic functions on U with zero at the point at infinity (if the latter belongs to U), endowed with the topology of uniform convergence on every compact subset of U (see ([1])). Finally, let $P_M(S_2 \setminus \{a\})$, for each $a \in M$, be the subspace of the topological vector space

$H(M, \{a\})$ consisting of the restrictions to M of all possible functions belonging to $P(S_2 \setminus \{a\})$.

Theorem 1. *For every finite subset $F = \{a_1, \dots, a_n\}$ in M, the space $H(M, F)$ is the topological direct sum of the subspaces $H(M)(= H(M, \emptyset))$ and $P_M(S_2 \setminus \{a_k\})$ $(k = 1, \dots, n)$.*

Corollary. *The space $H(M, \mathfrak{F})$ is the topological direct sum of its subspaces $H(M)$ and $P_M(S_2 \setminus \{a\})$ $(a \in M)$.*

Let M be a nonempty relatively compact open set in O. Then, for each $D \in \mathfrak{D}$, the intersection $D \cap M$ is finite; consequently, the restriction mappings

$$\pi_M : H(O, \mathfrak{D}) \to H(M, \mathfrak{F})$$

and

$$\rho_M : H(O, D) \to H(M, D \cap M)$$

are defined, and for them the diagram

$$\begin{array}{ccc} H(O, D) & \xrightarrow{\varphi_D} & H(O, \mathfrak{D}) \\ \rho_M \downarrow & & \downarrow \pi_M \\ H(M, D \cap M) & \xrightarrow{\psi_{D \cap M}} & H(M, \mathfrak{F}) \end{array}$$

commutes.

obviously, commutative. A direct verification shows that the topology of the space $H(O, D)$ is the weakest among all topologies for which the maps ρ_M are continuous. Hence it follows, in particular, that the maps π_M are continuous, since the maps $\pi_M \circ \varphi_D (= \psi_{D \cap M} \circ \rho_M)$ are continuous ((2), Ch. II, § 2, corollary of proposition 1).

Theorem 2. *The topology of the space $H(O, \mathfrak{D})$ is the weakest of all topologies for which the maps π_M are continuous (where M is an arbitrary relatively compact open set in O).*

Corollary 1. *For every $D \in \mathfrak{D}$ the topology of the space $H(O, D)$ coincides with that induced from $H(O, \mathfrak{D})$.*

Corollary 2. *The space $H(O, \mathfrak{D})$ is separated and complete.*

Theorem 3. *In order that a subset of the space $H(O, \mathfrak{D})$ be bounded, it is necessary and sufficient that it be contained in one of the spaces $H(O, D)$ $(D \in \mathfrak{D})$ and be bounded in it.*

Corollary 1. *A sequence of elements of the space $H(O, \mathfrak{D})$ is convergent if and only if it is contained in one of the spaces $H(O, D)$ $(D \in \mathfrak{D})$ and converges in it to the same limit.*

Corollary 2. *$H(O, \mathfrak{D})$ is a Montel space and, in particular, reflexive.*

Remark. One can show that $H(O,\mathfrak{D})$ is not a space of countable type and, consequently, is not metrizable (cf. ([3])).

2. Speaking of neighborhoods of a nonempty set A on the Riemann sphere, we shall assume that they are open and contain no connected components not meeting A. The spaces $P(U)$ (where U is an arbitrary neighborhood of the set $A \subset S_2$) and the restriction maps $P(U') \to P(U)$ $(U \subset U')$ form an inductive system. The inductive limit of this system $R(A)$ is canonically identified with the vector space of holomorphic functions, each of which is defined in some neighborhood of the set A; two such functions are regarded as equivalent (and represent one and the same element of the space $R(A)$) if they coincide in some neighborhood of the set A (cf. ([4])).

The vector space $H'(M)$, dual to $H(M)$ (where M is an arbitrary open subset of O), is canonically identified with the space $R(S_2 \setminus M)$ (see ([1],[5],[6])). In this case, for every $D \in \mathfrak{D}$, the vector space $H'(O,D)$ is identified with $R((S_2 \setminus O) \cup D)$, since there is a canonical isomorphism of the space $H(O,D)$ onto $H(O \setminus D)$. The vector spaces $R((S_2 \setminus O) \cup D)$ $(D \in \mathfrak{D})$ and the maps

$$R((S_2 \setminus O) \cup D') \to R((S_2 \setminus O) \cup D) \quad (D \subset D'),$$

dual to the embeddings $H(O,D) \to H(O,D')$, form a projective system. The projective limit $R(S_2 \setminus O, \mathfrak{D})$ of this system is canonically identified with the vector space $H'(O,\mathfrak{D})$, dual to $H(O,\mathfrak{D})$. Every continuous linear form on $H(O,\mathfrak{D})$ can be represented in the form

$$f \mapsto \int_\Gamma f(\zeta)\, g(\zeta)\, d\zeta,$$

where $f \in H(O,\mathfrak{D})$ (more precisely, $f \in H(O,D)$ for some $D \in \mathfrak{D}$), g is a holomorphic function in some neighborhood U of the set $(S_2 \setminus O) \cup D$ with zero at the point at infinity; Γ is the oriented boundary of some compact set in $O \setminus D$, consisting of paths in U. The map ${}^t\varphi_D$, dual to φ_D, is for every D the canonical projection assigning to an element of the space $R(S_2 \setminus O, \mathfrak{D})$ its representative in $R((S_2 \setminus O) \cup D)$.

Let M be a nonempty open subset of O and F a finite subset of M $(F \in \mathfrak{F})$. Then, by Theorem 1, the space

$$R(S_2 \setminus M, F) = R(S_2 \setminus M) \times \prod_{a \in F} R(\{a\})$$

canonically identifiable with the space dual to $H(M,F)$; similarly, by virtue of the corollary to Theorem 1, the space

$$R(S_2 \setminus M, \mathfrak{F}) = R(S_2 \setminus M) \times \prod_{a \in M} R(\{a\})$$

is canonically identifiable with the vector space $H'(M,\mathfrak{F})$, dual to $H(M,\mathfrak{F})$. Here the mapping ${}^t\psi_F$, dual to ψ_F, for each $F \in \mathfrak{F}$ sends a family of functions $g_0 \in R(S_2 \setminus M)$, $g_a \in R(\{a\})$ $(a \in M)$, which is an element of the space $R(S_2 \setminus M, \mathfrak{F})$, to the family g_0, g_a $(a \in F)$ as an element of the space $R(S_2 \setminus M, F)$.

The spaces $R(S_2 \setminus M, \mathfrak{F})$ (where M is an arbitrary relatively compact open set in O) and the mappings dual to the restriction mappings $H(M',\mathfrak{F}) \to H(M,\mathfrak{F})$ $(M \subset M')$, form an inductive system whose limit is canonically identifiable with the vector space $R(S_2 \setminus O, \mathfrak{D})$. The mapping ${}^t\pi_M$, dual to π_M, sends each family of functions $g_0 \in R(S_2 \setminus M)$, $g_a \in R(\{a\})$ $(a \in M)$ to a family of functions $h_D \in R((S_2 \setminus O) \cup D)$ $(D \in \mathfrak{D})$ in such a way that, for every $D \in \mathfrak{D}$, the function h_D coincides with g_0 in some neighborhood of the set $S_2 \setminus O$, and in some neighborhood of the point $a \in D$ this same function h_D coincides with g_a, if $a \in M$, and with g_0, if $a \in O \setminus M$.

Thus, every continuous linear form on $H(O,\mathfrak{D})$ is representable in the form

$$f \mapsto \int_{\Gamma_0} f_0(\xi) g_0(\xi)\, d\xi + \sum_{a \in M} \int_{\Gamma_a} f_a(\xi) g_a(\xi)\, d\xi,$$

where M is a relatively compact open set in O; f_a is the principal part of the Laurent expansion of the function $f \in H(O,\mathfrak{D})$ in a neighborhood of the point $a \in M$, and $f_0 = f - \sum_{a\in M} f_a$; g_0 and g_a $(a \in M)$ are functions holomorphic in some neighborhoods U_0 and U_a, respectively, of the sets $S_2 \setminus M$ and $\{a\}$ $(a \in M)$, with the point at infinity being a zero of g_0; finally, Γ_0 and Γ_a $(a \in M)$ are oriented boundaries of compact sets containing, respectively, M and $S_2 \setminus \{a\}$ $(a \in M)$, which consist of paths in U_0 and U_a $(a \in M)$, respectively.

3. The vector space $R(A)$, where A is an arbitrary nonempty subset of the Riemann sphere, shall be endowed with the strongest of the locally convex topologies for which the canonical mappings $P(U) \to R(A)$ are continuous (where U is an arbitrary neighborhood of the set A). Then the topological vector space $R((S_2 \setminus O) \cup D)$ is identifiable with the strong dual of the space $H(O,D)$ ((1), Proposition 16). The vector space $R(S_2 \setminus M, D \cap M)$, endowed with the product of the topologies of the spaces $R(S_2 \setminus M)$ and $R(\{a\})$ $(a \in D \cap M)$, is identifiable with the strong dual of the space $H(M, D \cap M)$. Further, the topology in $R(S_2 \setminus M, \mathfrak{F})$, being the weakest of the locally convex topologies for which the mappings $\psi_{D\cap M}$ $(D \in \mathfrak{D})$ are continuous and obviously coinciding with the product of the topologies of the spaces $R(S_2 \setminus M)$ and $R(\{a\})$ $(a \in M)$, is identifiable with the strong topology in the dual of $H(M,\mathfrak{F})$. Finally, by virtue of Mackey' s theorem:

Theorem 4. *The strongest of the locally convex topologies in $R(S_2 \setminus O, \mathfrak{D})$, for which the mappings ${}^t\pi_M$ are continuous (where M is an arbitrary relatively compact open set in O), is canonically identifiable with the strong topology in the space dual to $H(O,\mathfrak{D})$.*

Similarly, the topology of the space $R((S_2 \setminus O) \cup D)$ is the strongest of the locally convex topologies for which all mappings ${}^t\rho_M$ are continuous. From the commutative diagram

$$\begin{array}{ccc} R(S_2 \setminus M, \mathfrak{F}) & \xrightarrow{t_{\pi_M}} & R(S_2 \setminus O, \mathfrak{D}) \\ {\scriptstyle t_{\psi_{D\cap M}}} \downarrow & & \downarrow {\scriptstyle t_{\varphi_D}} \\ R(S_2 \setminus M, D \cap M) & \xrightarrow{t_{\rho_M}} & R((S_2 \setminus O) \cup D) \end{array}$$

it follows that the mappings t_{φ_D} $(D \in \mathfrak{D})$ are continuous.

Theorem 5. The topology of the space $R(S_2 \setminus O, \mathfrak{D})$ is the weakest among all topologies for which the mappings t_{φ_D} $(D \in \mathfrak{D})$ are continuous.

Theorem 6. In order that a subset of the space $R(S_2 \setminus O, \mathfrak{D})$ be bounded, it is necessary and sufficient that it be the image under the mapping t_{π_M}, for some relatively compact open $M \subset O$, of a bounded set from $R(S_2 \setminus M, \mathfrak{F})$.

An analogous proposition also holds for the space $R((S_2 \setminus O) \cup D)$. Moreover, for any $D \in \mathfrak{D}$ the mapping t_{φ_D} is an epimorphism, and every bounded set in $R((S_2 \setminus O) \cup D)$ is the image under t_{φ_D} of some bounded set from $R(S_2 \setminus O, \mathfrak{D})$.

4. In conclusion we shall explain why the space $H(O, \mathfrak{D})$ cannot be embedded in the space $H(O, \mathfrak{S})$ of holomorphic functions on O with a finite number of non-isolated singularities.

Let $\mathfrak{S}$ be the set of all closed subsets in O, each of which has only a finite number of non-isolated points, and let $H(O, S)$, for each $S \in \mathfrak{S}$, be the vector space of all holomorphic functions on O with singularities in S, endowed with the topology of uniform convergence on each compact subset of $O \setminus S$. The set $\mathfrak{S}$ is filtered by inclusion, $H(O, S) \subset H(O, S')$ for $S \subset S'$, and the topology in $H(O, S)$ coincides with that induced from $H(O, S')$. The vector space $H(O, \mathfrak{S})$, which is the union of the subspaces $H(O, S)$ $(S \in \mathfrak{S})$, is naturally endowed with the strongest of the locally convex topologies for which the canonical mappings $H(O, S) \to H(O, \mathfrak{S})$ $(S \in \mathfrak{S})$ are continuous. It turns out, however, that the topology in $H(O, \mathfrak{S})$ so defined is trivial: its only neighborhood of zero coincides with the whole space.

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Note: Figure translations are in progress. See original paper for figures.