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Abstract

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A. L. KRYLOV, V. G. SHUSTER

ON THE STABILITY OF THE CENTRAL POSITION OF A JOURNAL IN A HYDRODYNAMIC BEARING

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It is customary to study the motion of a viscous fluid (gas) in narrow regions in the approximation of lubrication theory, whose equations were first written down by Reynolds in 1886. For the Reynolds equations a number of stationary problems have been solved (see, for example, [1] for liquid lubrication and [2] for gas lubrication).

Recently, much attention has been attracted by questions of the stability of bearing operation, considered within the framework of lubrication theory ([3,4]). The relation between stationary solutions of the Reynolds equations (lubrication theory) and the Navier–Stokes equations was studied in ([5]) (for slow flows and a wide clearance) and ([6]) (for a narrow clearance). In these works it was shown that, under natural assumptions, solutions of the Navier–Stokes equations are close to solutions of the lubrication equations.

In the present work, in a hydrodynamic formulation, the question of the stability of the journal in the central position is studied. Mathematically the problem is formulated as follows. A viscous incompressible fluid (lubricant) fills the space between a fixed outer cylinder (the bearing) and a movable inner one (the journal). The journal rotates with a prescribed angular velocity ω and moves in the lubricant under the action of hydrodynamic forces and an external force $\mathbf{W}_0$ applied to its center. In particular, when $\mathbf{W}_0 = 0$, a stationary regime is possible, characterized by a coaxial arrangement of the journal and the bearing. Below the stability of this regime with respect to infinitesimally small perturbations is investigated. It turns out that this regime is unstable and, in the linear approximation, "breaks down" into motion along a slowly unwinding spiral with the angular velocity of the center of the journal equal to $\omega/2$. This result is obtained under the additional assumption of smallness of the clearance between the journal and the bearing.

We consider the equations

$$-M\ddot{\mathbf{y}} + \mathbf{W}_0 + \oint_C \mathbf{t}\,dl = 0; \tag{1}$$

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\frac{p}{\rho_0} + 2\nu\operatorname{div}D; \tag{2}$$

$$\operatorname{div}\mathbf{v} = 0. \tag{3}$$

Boundary conditions:

$$\mathbf{v}|_C = \mathbf{v}_C = \dot{\mathbf{y}} + \omega|\mathbf{x}_C - \mathbf{y}|\,\vec{\tau}; \qquad \mathbf{v}|_{C'} = \mathbf{v}_{C'} = 0,$$

where (1) is the equation of motion of the center of the journal; (2) and (3) are the equations of motion of the fluid in the variable region between the cylinders C and C'; $\mathbf{y}$, $\mathbf{x}_C$, and $\mathbf{x}_{C'}$ see Fig. 1; $\mathbf{t} = -p\mathbf{n} + 2\nu\rho_0\mathbf{n}D$ is the stress vector of the fluid; $\mathbf{v}$ is the velocity vector of motion of the fluid particles; D is the deformation tensor, in mixed components it is written in the form $D_k^i = {}^1/_2(\nabla_k v^i + \nabla^i v_k)$, in particular, in a Cartesian system $(\operatorname{div}D)_k = \Delta v_k$; p is the pressure in the fluid; ν is the kinematic viscosity of the fluid; ρ_0 is the density of the fluid; $M = \pi R^2\rho_1$ is the mass of the journal; ρ_1 is the density of the journal material; $\vec{\tau}$ is the unit vector tangent to the contour C at the point $\mathbf{x}_C$; $|\mathbf{x}_C - \mathbf{y}| = R$;

$$|x_{C'}| = R + \delta; \quad \delta = R_1 - R$$

is the radial clearance between the journal and the bearing (see Fig. 1).

Let us consider the stability of the central stationary position of the journal, characterized by the conditions:

$$\mathbf{W}_0 = 0; \quad \mathbf{y}_0 = 0; \quad \mathbf{v} = \mathbf{v}_0; \quad p = p_0, \text{ etc.}$$

Suppose the journal has received a small displacement $\delta\mathbf{y}$ from the central position; denote

$$\mathbf{y} = \delta\mathbf{y}; \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1; \quad p = p_0 + p_1; \quad \mathbf{t} = \mathbf{t}_0 + \mathbf{t}_1; \quad x_{C'0} = x_{C'1};$$

$$x_{C1} = x_{C0} + \delta\mathbf{y},$$

where quantities with subscript 0 refer to the stationary solution; $C0$ is the stationary position of the contour C; $C1$ is the perturbed position of the contour.

We shall linearize the equations of motion (1)–(3) (taking account of the mobility of the boundary); we write the resulting system in dimensionless form. We obtain

Fig. 1. Geometrical diagram of a hydrodynamic bearing

Figure 1: Fig. 1. Geometrical diagram of a hydrodynamic bearing

$$\delta\ddot{\mathbf{y}}_\sigma + \frac{\varkappa}{\pi\gamma\lambda}\left(\oint_{C0\sigma} \mathbf{t}_{1\sigma}\,dl + \oint_{C1\sigma} \mathbf{t}_{0\sigma}\,dl\right) = 0, \tag{4}$$

$$\frac{\partial \mathbf{v}_{1\sigma}}{\partial t_\sigma} + \mathbf{v}_{0\sigma}\,\mathrm{grad}\,\mathbf{v}_{1\sigma} + \mathbf{v}_{1\sigma}\,\mathrm{grad}\,\mathbf{v}_{0\sigma} =$$

$$= -\varkappa\,\mathrm{grad}\,p_{1\sigma} + \frac{2}{\chi}\,\mathrm{div}\,D_{1\sigma},$$

$$\mathrm{div}\,\mathbf{v}_{1\sigma} = 0$$

with boundary conditions in the region between $C0\sigma$ and $C'0\sigma$:

$$\mathbf{v}_{1C0\sigma} = \lambda\delta\dot{\mathbf{y}}_\sigma + 2\lambda b(\mathbf{n}, \delta\mathbf{y}_\sigma)\vec{\tau} - \lambda\delta\mathbf{y}_\sigma F,$$

$$\mathbf{v}_{1C'0\sigma} = 0. \tag{5}$$

Fig. 1. Geometrical diagram of a hydrodynamic bearing

The index σ, indicating that all quantities are dimensionless, will be omitted in all subsequent calculations.

The relations between dimensionless and dimensional quantities have the form:

$$t_\sigma = \omega t; \quad \mathbf{t}_\sigma = \frac{1}{p_a}\mathbf{t},$$

where p_a is atmospheric pressure;

$$\delta\mathbf{y}_\sigma = \frac{1}{\delta}\delta\mathbf{y};$$

the contours $C0\sigma$ and $C1\sigma$ are circles of unit radius: the contour $C'0\sigma$ is a circle of radius $(1+\lambda)$;

$$\mathbf{x}_{C0\sigma} = \mathbf{n}; \quad \mathbf{x}_{C1\sigma} = \mathbf{n} + \lambda\delta\mathbf{y}_\sigma;$$

$$\mathbf{v}_{0\sigma} = \frac{1}{\omega R}\mathbf{v}_0; \quad \mathbf{v}_{1\sigma} = \frac{1}{\omega R}\mathbf{v}_1; \quad p_{1\sigma} = \frac{1}{p_a}p_1; \quad D_{1\sigma} = \frac{1}{\omega}D_1.$$

The dimensionless parameters are:

$\lambda = \delta/R$; $\varkappa = p_a/\rho_0\omega^2R^2$; $\chi = \omega R^2/\nu$; $\gamma = \rho_1/\rho_0$; $b = (1+\lambda)^2/\lambda(2+\lambda)$.

The tensor grad $\mathbf{v}$ in mixed components is the covariant derivative of the contravariant velocity vector. The vector

$$\mathbf{H} = \mathbf{v}_0 \operatorname{grad} \mathbf{v}_1$$

in covariant components is

$$H_i = v_0^k \nabla_k v_1^i.$$

The vector

$$\mathbf{P} = \operatorname{div} D$$

in covariant components is

$$P_i = \nabla_k(\nabla_i v^k + \nabla^k v_i).$$

F is the rotation tensor; in curvilinear orthogonal coordinates, in physical components

$$F_{11} = F_{22} = 0, \quad F_{12} = 1, \quad F_{21} = -1.$$

Integrating the third equation of system (4), we obtain:

$$\mathbf{v}_1 = \operatorname{rot} \psi,$$

where

$$\vec{\psi} = \psi \mathbf{k}$$

($\mathbf{k}$ is the unit vector normal to the plane of rotation, ψ is the stream function).

Using the boundary conditions (5) and the stationary solution

$$\mathbf{v}_0 = (a + b/x^2)\mathbf{x}F, \quad \text{where } a = -1/\lambda(2+\lambda),$$

and introducing the polar coordinate system (r, θ) with the pole at the center of the bearing, we reduce system (4) to the form

$$\delta\ddot{\mathbf{y}} - 6\,\frac{1}{(\gamma-1)\chi}\,b\,\delta\mathbf{y}F - \frac{1}{\pi(\gamma-1)\chi\lambda}\oint_{C0}\frac{\partial^3\psi}{\partial r^3}\vec{\tau}\,dl = 0. \tag{6}$$

$$\frac{\partial\Delta\psi}{\partial\hat{t}} = -\omega(r)\frac{\partial\Delta\psi}{\partial\theta} + \frac{1}{\chi}\Delta^2\psi, \quad \text{where } \omega(r) = a + \frac{b}{r^2}, \tag{7}$$

with boundary conditions

$$\operatorname{rot}\,\vec{\psi}_{C0} = \lambda\dot{\delta y} + 2\lambda b(\mathbf{n}, \delta\mathbf{y})\,\vec{\tau} - \lambda\dot{\delta y}F, \tag{8}$$

$$\text{rot}\ \vec{\psi}_{C'0} = 0.$$

In the linear equation (7) we separate the variables. Representing the function ψ in the form

$$\psi = \sum_{-\infty}^{+\infty} e^{\sigma_n t} f_n(r) e^{in\theta}, \tag{9}$$

we reduce equation (7) to a system of ordinary differential equations:

$$f_n^{(\text{IV})} + \frac{2}{r} f_n''' - \left[\chi(\sigma_n + in\omega(r)) + \frac{1+2n^2}{r^2}\right] f_n'' - \frac{1}{r}\left[\chi(\sigma_n + in\omega(r)) - \frac{1+2n^2}{r^2}\right] f_n' + \frac{1}{r^2}\left[\chi(\sigma_n + in\omega(r)) - n^2 \frac{4-r}{r^2}\right.$$

Reducing the interval of integration to $[0, 1]$ by the linear substitution $r = \lambda r_1 + 1$, we obtain

$$\begin{aligned} f_n^{\text{V}} &+ \lambda \frac{2}{\lambda r_1 + 1} f_n''' - \lambda^2 \left[\chi(\sigma_n + in\omega(\lambda r_1 + 1)) + \frac{1+2n^2}{(\lambda r_1 + 1)^2}\right] f_n'' \\ &- \lambda^3 \frac{1}{\lambda r_1 + 1}\left[\chi(\sigma_n + in\omega(\lambda r_1 + 1)) - \frac{1+2n^2}{(\lambda r_1 + 1)^2}\right] f_n' \\ &+ \lambda^4 \frac{1}{(\lambda r_1 + 1)^2}\left[\chi(\sigma_n + in\omega(\lambda r_1 + 1)) - n^2 \frac{4-n^2}{(\lambda r_1 + 1)^2}\right] f_n = 0. \end{aligned} \tag{10}$$

Equation (6), taking into account the boundary conditions (8), has the solution

$$\delta \mathbf{y} = \sum_{-\infty}^{+\infty} (E_n \mathbf{i} + B_n \mathbf{j}) e^{\sigma_n t}, \tag{11}$$

where $\mathbf{i}, \mathbf{j}$ are constant unit vectors; $\mathbf{j} = i\mathbf{i}'$, the vector $\mathbf{i}$ is directed along the polar axis; E_n, B_n are constants depending on σ_n.

Substituting expressions (9) and (11) into the boundary conditions (8), we find that for all n, except $n = \pm 1$, $f_n(0) = f_n'(0) = 0$; and since $f_n(1) = f_n'(1) = 0$ for all n and equations (10) are homogeneous, it follows that $f_n(r_1) \equiv 0$ for $n \neq \pm 1$.

For $n = \pm 1$, from equation (6) and the boundary conditions (8) we obtain the boundary conditions for equation (10):

$$f_\pm(0) = L_1^\pm(\sigma_\pm) f_\pm'''(0), \qquad f_\pm'(0) = L_2^\pm(\sigma_\pm) f_\pm'''(0), \qquad f_\pm(1) = 0, \qquad f_\pm'(1) = 0, \tag{12}$$

where

$$f_+(r_1),\ L_1^+(\sigma_+),\ L_2^+(\sigma_+) \quad \text{correspond to } n=+1;$$

$$f_-(r_1),\ L_1^-(\sigma_-),\ L_2^-(\sigma_-) \quad \text{correspond to } n=-1;$$

$$L_1^\pm(\sigma_\pm) = -\frac{a_1}{\lambda^3}(1 \mp i\sigma_\pm)\frac{1}{a_2/\lambda \mp i\sigma_\pm^2}; \qquad L_2^\pm(\sigma_\pm) = \frac{a_1}{\lambda^3}(\xi \pm i\lambda\sigma_\pm)\frac{1}{a_2/\lambda \mp i\sigma_\pm^2}.$$

$$a_1 = \frac{1}{(\gamma-1)\chi}; \qquad a_2 = 6a_1\frac{(1+\lambda)^2}{2+\lambda}; \qquad \xi = \frac{1+(1+\lambda)^2}{2+\lambda}.$$

The investigation of the solutions of (10) as $\lambda \to 0$ reduces to the investigation of two equations

$$f_\pm^{(\mathrm{IV})}(r_1) = 0 \tag{13}$$

with boundary conditions (12).

In order that system (12)–(13) have a nontrivial solution, its characteristic determinant must be equal to zero. This leads us to the equations

$$\sigma_+^2 + \frac{12a_1}{\lambda^3}\sigma_+ + i\,\frac{6a_1}{\lambda^3} = 0 \quad \text{for } n=+1,$$

$$\sigma_-^2 + \frac{12a_1}{\lambda^3}\sigma_- - i\,\frac{6a_1}{\lambda^3} = 0 \quad \text{for } n=-1.$$

Each of these equations has one bounded root as $\lambda \to 0$:

$$\sigma_+ = \frac{1}{48}\frac{\lambda^3}{a_1} - \frac{i}{2} + O(\lambda^6) \quad \text{for } n=+1,$$

$$\sigma_- = \frac{1}{48}\frac{\lambda^3}{a_1} + \frac{i}{2} + O(\lambda^6) \quad \text{for } n=-1.$$

It follows from this:

1. The central position of the journal under investigation is unstable ($\mathrm{Re}\,\sigma_\pm > 0$).

2. The loss of stability leads to a motion which, in the linear approximation, is a rotation of the center of the journal about the center of the bearing with angular velocity $\omega/2$ $\left(\operatorname{Im}\sigma_\pm = \mp\frac{1}{2}\right)$ and a slow unwinding from the center $(\operatorname{Re}\sigma_\pm \sim \lambda^3 > 0)$.

The problem considered is a special case of the more general problem of the stability of a general stationary position of a journal in a bearing with an arbitrary clearance (i.e., with any λ). There are grounds to believe that in the general case the stationary position of the journal will prove unstable, and moreover that the instability will grow as the clearance increases. Therefore the restrictions imposed in the formulation of the problem (the central stationary position of the journal and the smallness of λ) are not related to the essence of the question of the stability of the journal and were introduced only to simplify the problem. The question of the motion of the journal after loss of stability reduces to consideration of a nonstationary problem in the full nonlinear formulation and is not touched upon in the present work.

Moscow State University
named after M. V. Lomonosov

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