ON THE DUALS OF CONSTRUCTIVE LOCALLY CONVEX SPACES

MATHEMATICS

1966

SovietRxiv

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UDC 517.11

MATHEMATICS

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ON THE DUALS OF CONSTRUCTIVE LOCALLY CONVEX SPACES

(Presented by Academician A. A. Dorodnitsyn on V 6, 1965)

In the present note we use the terms and notation introduced in ([1–3,7,8]). In ([7]) the concept of a constructive locally convex space was introduced. Let $\mathfrak{M}$ be a constructive locally convex space, given by the list:

$$A_{\mathfrak{m}},\ \mathfrak{P},\ \mathfrak{C},\ +,\ \cdot,\ \mathfrak{D},\ A_n,\ \mathfrak{I},\ \mathfrak{D}. \tag{1}$$

The notation θ, ι will be understood in the same way as in ([7]). As defined in ([8]), a set $\mathfrak{A}$ of words of type θ is called **bounded** in the space $\mathfrak{M}$ if:

$$\forall l_1 \exists a\, (a > 0\ \&\ \forall \theta_1 (\theta_1 \in \mathfrak{A} \supset \mathfrak{D}(l_1, a \cdot \theta_1))).$$

Denote by $\langle P \rangle$ an algorithm in the alphabet $A_{\mathfrak{m}}^{ca} \cup {}_2$, whose notation is the word P. Introduce the notation:

$$\mathfrak{K} \Longleftrightarrow ((\langle c_{0,1} \rangle \in (\iota \to p))\ \&\ \forall \iota_1 (\langle c_{0,1} \rangle (\iota_1) > 0)),$$

where $c_{0,1}$ is a variable for words in ${}_0$. Since $\mathfrak{I}$ is a normal formula, $\mathfrak{K}$ is also a normal formula. Denote by $\mathfrak{x}$ the subordinate generic letter whose characteristic formula is $\mathfrak{K}$.

Let: A_r be an alphabet; $\mathfrak{X}$ a one-parameter formula in a variable of genus t_r; $\mathfrak{R}$ a formula in variables ε of genus t_r and η of genus θ. Denote by $\mathfrak{v}$ the subordinate generic letter whose characteristic formula is $\mathfrak{X}$. Introduce the notation

$$\mathfrak{R}(V, U) \Longleftrightarrow F_{V,U}^{\varepsilon,\eta} \mathfrak{R}_{\perp},$$

where V is any term of genus $\mathfrak{v}$; U is any term of genus θ.

We shall call the list $(A_r, \mathfrak{X}, \mathfrak{R})$ a **fundamental system of bounded sets** in the space $\mathfrak{M}$, if the conditions

$$\forall v_1 \exists \mathfrak{x}_1 \forall \theta_1 (\mathfrak{R}(v_1, \theta_1) \supset \forall \iota_1 \mathfrak{D}(\iota_1, \langle \mathfrak{x}_1 \rangle (\iota_1) \cdot \theta_1));$$

$$\forall \mathfrak{x}_1 \exists v_1 \forall \theta_1 (\forall \iota_1 \mathfrak{D}(\iota_1, \langle \mathfrak{x}_1 \rangle (\iota_1) \cdot \theta_1) \supset \mathfrak{R}(v_1, \theta_1)).$$

are satisfied.

A fundamental system of bounded sets $(A_r, \mathfrak{X}, \mathfrak{R})$ is called **normal** if $\mathfrak{X}$ is a normal formula.

Theorem 1. *For any constructive locally convex space one can construct in it a normal fundamental system of bounded sets.*

This is the system $({}_0, \mathfrak{K}, \mathfrak{R})$, where the formula $\mathfrak{R}$ is defined so that

$$\mathfrak{R}(\mathfrak{x}_1, \theta_1) \equiv \forall \iota_1 \mathfrak{D}(\iota_1, \langle \mathfrak{x}_1 \rangle (\iota_1) \cdot \theta_1).$$

2. Let $A_\eta \Longleftrightarrow A_\mathfrak{m} \cup {}_3$. An algorithm λ in A_η^{ca} is called a **functional** in the space $\mathfrak{M}$, if it is an operator of type $(\theta \to \mathbb{D})$ (see ([7])).

Let λ_1 be a linear functional in $\mathfrak{M}$. λ_1 is continuous if

$$\forall n \exists \iota_1 \forall \theta_1 (\mathfrak{D}(\iota_1, \theta_1) \supset M(\lambda_1(\theta_1)) < 2^{-n}).$$

This means that there is realizable in A_ξ^{ca} $(A_\xi \Longleftrightarrow A_\eta \cup {}_0)$ an algorithm λ_2 of type

$(\mathbf{n} \to \iota)$ such that

$$\forall n \theta_1 (\mathfrak{P}(\lambda_2(n), \theta_1) \supset M(\lambda_1(\theta_1) < 2^{-n})). \tag{2}$$

An algorithm λ_2 of type $(\mathbf{n} \to \iota)$ satisfying (2) will be called a **continuity regulator** of the functional λ_1.

A linear functional λ_1 in $\mathfrak{M}$ is called **quasicontinuous** if it cannot be discontinuous.

Theorem 2. *There exists a constructive locally convex space for which it is false that every quasicontinuous linear functional is continuous.*

Theorem 3. *There exists a constructive locally convex space such that no algorithm is possible which, for each continuous linear functional in it, enables one to find a continuity regulator of this functional.*

3. We shall say of a word $t_{\pi,1}$ in the alphabet $A_\pi(\rightleftarrows {}_0 \cup \{\odot\})$ that it is a **complete cipher of a linear discontinuous functional** in $\mathfrak{M}$, and we shall write

$$\left(t_{\pi,1} \in \frac{\text{compl. disc.}}{\mathfrak{m}}\right), \tag{3}$$

if the word $t_{\pi,1}$ has the form $P \odot Q$, where P is a record of a linear discontinuous functional in $\mathfrak{M}$ and Q is a record of its continuity regulator. Let

$$t_{\pi,1} \doteq P \odot Q.$$

We shall denote:

$$t_{\pi,1} \rightleftarrows \langle P \rangle_\eta; \qquad t_{\pi,1} \rightleftarrows \langle Q \rangle_\zeta.$$

Denote by $\mathfrak{P}^{\mathfrak{m}}$ formula (3), and by $\mathfrak{m}$ the generic letter whose characteristic formula is $\mathfrak{P}^{\mathfrak{m}}$. Introduce the formula

$$\mathfrak{C}^{\mathfrak{m}} \rightleftarrows \forall \theta_1 (t_{\pi,1}(\theta_1) \overset{\text{B}}{=} t_{\pi,2}(\theta_1)).$$

Denote by $\dot{\mathfrak{m}}^+$ an algorithm of type $(\mathfrak{m}\mathfrak{m} \to \mathfrak{m})$, and by $\dot{\mathfrak{m}}$ an algorithm of type $(p\mathfrak{m} \to \mathfrak{m})$, such that

$$\forall \mathfrak{m}_1 \mathfrak{m}_2 \theta_1 (\dot{\mathfrak{m}}^+(\mathfrak{m}_1 \square \mathfrak{m}_2)(\theta_1) \overset{\text{B}}{=} \mathfrak{m}_1(\theta_1) + \mathfrak{m}_2(\theta_1));$$

$$\forall a \mathfrak{m}_1 \theta_1 (\dot{\mathfrak{m}}(a \square \mathfrak{m}_1)(\theta_1) \overset{\text{B}}{=} a \cdot \mathfrak{m}_1(\theta_1)).$$

Denote by $\mathfrak{D}^{\mathfrak{m}}$ the word $U \odot V$, where U is a record of an algorithm transforming each word of type θ into the number 0; V is a record of an algorithm transforming each number n into a fixed word of type ι.

Let $(A_r, \mathfrak{X}, \mathfrak{R})$ be a normal fundamental system of bounded sets in $\mathfrak{M}$. Construct a formula $\mathfrak{D}^{\mathfrak{m}}$ such that, whatever the word ν_1 of type ν and the word $\mathfrak{m}_1$ of type $\mathfrak{m}$, one has

$$\mathfrak{D}^{\mathfrak{m}}(\nu_1, \mathfrak{m}_1) \equiv \forall \theta_1 (\mathfrak{R}(\nu_1, \theta_1) \supset M(\mathfrak{m}_1(\theta_1)) \leq 1).$$

It is not hard to prove that the list

$$A_\pi,\ \mathfrak{P}^{\mathfrak{m}},\ \mathfrak{C}^{\mathfrak{m}},\ \mathfrak{m}^+,\ \dot{\mathfrak{m}},\ \mathfrak{D}^{\mathfrak{m}},\ A_r,\ \mathfrak{X},\ \mathfrak{D}^{\mathfrak{m}}$$

is a constructive locally convex space, which we shall call **conjugate to the space $\mathfrak{M}$**.

4. Let $\mathfrak{M}$ be a normed space. It may be regarded as a locally convex space. Consequently, one can construct the space conjugate to the normed space $\mathfrak{M}$. This conjugate is a locally convex space.

Theorem 4. *Let $\mathfrak{M}$ be a normed space with multiplication by real duplets, and let N be its norm. If the set of points θ_1 such that $N(\theta_1) = 1$ is compact* *then the space conjugate to $\mathfrak{M}$ is normable**.*

* See the definition in ([3]), § 11.
** See the definition in ([8]).

Theorem 5. There exists a constructive normed space such that its conjugate is not normable (not even seminormable) (see the definition in (8)′).

Theorem 6. There exists a constructive multinormed space such that its conjugate is not multinormable (see (8)).

Theorem 7. There exists a constructive complete Hilbert space such that it is not isomorphic to its conjugate.

For the proof of Theorems 5–7 it suffices to consider the space l_2, constructed in (3), § 13.

Theorem 8. For natural numbers m, n, $0 \leqslant m \leqslant n$, there exists a T'-separable, n-dimensional in the strong sense locally convex space such that its conjugate is an m-dimensional, normable space.

Theorem 9. For any natural number m there exists a T'-separable, infinite-dimensional locally convex space such that its conjugate is an m-dimensional, normable space.

Theorems 8 and 9 are proved with the aid of the space $\mathfrak{S}$, constructed in (7).

6. We shall consider bounded sets in the inductive limit of a sequence of locally convex spaces.

Let A_m and A_n be alphabets; $\mathfrak{P}$ a two-parameter formula in the variables k and α, where k is a variable for natural numbers and α is a variable of type t_m; $\mathfrak{E}$ a three-parameter formula in the variables k and β, γ of type t_m; $+$ an algorithm of type $(t_m t_m \dot{\to} t_m)$; $\cdot$ an algorithm of type $(p t_m \dot{\to} t_m)$; $\mathfrak{D}$ a fixed word of type t_m; $\mathfrak{I}$ a normal two-parameter formula in the variables k and δ of type t_n; $\mathfrak{L}$ a three-parameter formula in the variables k, ξ of type t_n and η of type t_m.

We agree to denote:

$$\mathfrak{P}(N,R) \Leftrightarrow F^{k,\alpha}_{N,R}\,\mathfrak{P}_{\perp}; \qquad \mathfrak{E}(N,R,S) \Leftrightarrow F^{k,\beta,\gamma}_{N,R,S}\,\mathfrak{E}_{\perp}; \qquad (R+S) \Leftrightarrow +(R\square S);$$

$$(a \cdot R) \Leftrightarrow \cdot(a\square R); \qquad \mathfrak{I}(N,T) \Leftrightarrow F^{k,\delta}_{N,T}\,\mathfrak{I}_{\perp}; \qquad \mathfrak{L}(N,T,R) \Leftrightarrow F^{k,\xi,\eta}_{N,T,R}\,\mathfrak{L}_{\perp},$$

where N is any natural number, R and S are any terms of type t_m, T is any term of type t_n, and a is any rational number. Suppose that

$$\forall k\alpha(\mathfrak{P}(k,\alpha) \supset \mathfrak{P}(k+1,\alpha)); \qquad \forall k\,\mathfrak{P}(k,\mathfrak{D});$$

$$\forall k\beta\gamma(\mathfrak{P}(k,\beta)\&\mathfrak{P}(k,\gamma) \supset (\mathfrak{E}(k,\beta,\gamma) \equiv \mathfrak{E}(k+1,\beta,\gamma)));$$

$$\forall k\beta\gamma(\mathfrak{P}(k+1,\beta)\&\mathfrak{P}(k,\gamma)\&\mathfrak{E}(k+1,\beta,\gamma) \supset \mathfrak{P}(k,\beta));$$

$$\forall k\beta\gamma(\mathfrak{P}(k,\beta)\&\mathfrak{P}(k,\gamma) \supset !(\beta+\gamma)\&\mathfrak{P}(k,\beta+\gamma));$$

$$\forall ka\alpha(\mathfrak{P}(k,\alpha) \supset !(a\cdot\alpha)\&\mathfrak{P}(k,a\cdot\alpha)).$$

Under these assumptions we shall call the list

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D},\ A_n,\ \mathfrak{I},\ \mathfrak{L} \tag{4}$$

an **expanding sequence of constructive locally convex spaces**, if for each fixed natural number k this list represents a constructive locally convex space (we denote it by $\mathfrak{M}_k$) and if the identity operator from each $\mathfrak{M}_k$ into $\mathfrak{M}_{k+1}$ is continuous (see the definition of a continuous operator in (7)).

Let the list (4) be an expanding sequence of locally convex spaces. Introduce the notation:

$$\mathfrak{P}^* \Leftrightarrow \exists k\,\mathfrak{P}(k,\alpha);$$

$$\mathfrak{E}^* \Leftrightarrow \exists k(\mathfrak{P}(k,\beta)\&\mathfrak{P}(k,\gamma)\&\mathfrak{E}(k,\beta,\gamma));$$

$$\mathfrak{I}^* \Leftrightarrow (\langle c_{0,1}\rangle_s \in (\mathrm{H} \to t_n))\&\forall k\,\mathfrak{I}(k,\langle c_{0,1}\rangle_s(k));$$

$$\mathfrak{L}^* \Leftrightarrow \exists c_{0,2} m\Big((\langle c_{0,2}\rangle_r \in (\mathrm{H} \to t_m))\&\forall k(\mathfrak{P}(k,\langle c_{0,2}\rangle_r(k))\&$$

$$\&\mathfrak{L}(k,\langle c_{0,1}\rangle_s(k),\langle c_{0,2}\rangle_r(k)))\&F^{\beta,\gamma}_{\alpha,\Sigma(m\Box c_{0,2})}\,\mathfrak{E}^*\Big),$$

where s is the number of the alphabet $A_n \cup \mathcal{U}_0$, r is the number of the alphabet $A_m \cup \mathcal{U}_0$, and Σ is an algorithm in A_r^α having the following property:

$$\Sigma(m\square c_{0,2}) \simeq \langle c_{0,2}\rangle_r(0) + \dots + \langle c_{0,2}\rangle_r(m).$$

It is not difficult to prove that the list

$$A_m,\ \mathfrak{P}^*,\ \mathfrak{C}^*,\ +,\ \cdot,\ \mathcal{O},\ \mathcal{U}_0,\ \mathfrak{I}^*,\ \mathcal{O}^* \tag{5}$$

is a locally convex space. We shall call it the **inductive limit of the sequence of spaces $\mathfrak{M}_k$**.

We shall say that:

1) The inductive limit (5) is α-regular if, for every bounded set $\mathfrak{A}$ in (5), there is a realizable number k such that $\mathfrak{A}$ is contained in $\mathfrak{M}_k$.
2) The inductive limit (5) is β-regular if, for every set $\mathfrak{A}$ bounded in (5) and contained in $\mathfrak{M}_k$, there is a realizable number j such that $\mathfrak{A}$ is bounded in $\mathfrak{M}_j$.
3) The inductive limit (5) is **regular** if it is α-regular and β-regular (cf. the classical definition in ([5])).

We shall say that in the sequence (4) each $\mathfrak{M}_k$ is a subspace of the space $\mathfrak{M}_{k+1}$ if the following conditions are satisfied:

$$\forall kj\delta(\mathfrak{I}(k,\delta) \equiv \mathfrak{I}(j,\delta));$$

$$\forall k\delta\alpha(\mathfrak{I}(k,\delta)\ \&\ \mathfrak{P}(k,\alpha) \supset (\mathcal{O}(k,\delta,\alpha) \equiv \mathcal{O}(k+1,\delta,\alpha))).$$

The inductive limit (5) is called **strict** if, in the sequence (4), each $\mathfrak{M}_k$ is a Z-closed subspace of $\mathfrak{M}_j$ for $j > k$.

The theorem on the regularity of strict inductive limits in classical mathematics ([6]) carries over to constructive mathematics in the following form:

Theorem 10. *Let $\mathfrak{M}$ be a strict inductive limit of a sequence of constructive locally convex spaces $\mathfrak{M}_k$. Then:*

a) *if $\mathfrak{A}$ is a bounded set in $\mathfrak{M}$, then*

$$\neg\forall k\, \mathfrak{N}_1((\theta_1 \in \mathfrak{A})\ \&\ \neg(\theta_1 \in \mathfrak{M}_k)).$$

b) *$\mathfrak{M}$ is β-regular.*

On the other hand, one can prove the following theorem:

Theorem 11. *There exists a strict inductive limit $\mathfrak{M}$ of a sequence of constructive locally convex spaces $\mathfrak{M}_k$ such that: a) there is no algorithm which, for each bounded set $\mathfrak{A}$ in $\mathfrak{M}$, makes it possible to find a number k such that $\mathfrak{A}$ is* *contained in $\mathfrak{M}_k$; b) it is false that for every bounded set $\mathfrak{A}$ in $\mathfrak{M}$ there is a least number k such that $\mathfrak{A}$ is contained in $\mathfrak{M}_k$.*

Part a) is proved with the aid of the theorem stating that there exists an algorithm of type $(\mathbb{N} \to \mathbb{N})$ whose applicability problem is not decidable ([1,3]); part b) is proved with the aid of Corollary 1 of Theorem 1 in ([4]).

In conclusion the author expresses his deep gratitude to A. A. Markov for a number of valuable comments and suggestions.

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Received
23 IV 1965

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Note: Figure translations are in progress. See original paper for figures.