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Abstract

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MATHEMATICS

E. S. Shtatland

ASYMPTOTICS OF THE DISTRIBUTION OF THE MAXIMUM FOR A CERTAIN CLASS OF PROCESSES WITH INDEPENDENT INCREMENTS

(Presented by Academician Yu. V. Linnik on 13 I 1966)

Let $\xi(t)$ be a homogeneous process with independent increments, having only positive jumps and a negative drift. Then ([1])

$$\mathbf{M}e^{-s\xi(t)} = e^{tA(s)} \qquad (s = \sigma + i\tau,\ \sigma > 0),$$

where

$$A(s) = \beta s + \int_0^\infty \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x), \quad -\beta - \int_0^\infty \frac{x}{1+x^2}\, dN(x) < 0.$$

Introduce the notation

$$F(t,x) = \mathbf{P}\left\{\sup_{0<u\le t} \xi(u) < x\right\}; \qquad F(x) = \mathbf{P}\left\{\sup_{0\le t<\infty} \xi(t) < x\right\};$$

$$\varphi(z,x) = \int_0^\infty e^{-zt} F(t,x)\, dt \qquad (\operatorname{Re} z > 0);$$

$$\bar\varphi(z,s) = \int_0^\infty \int_0^\infty e^{-zt-sx} F(t,x)\, dt\, dx \qquad (\operatorname{Re} z > 0,\ \operatorname{Re} s > 0).$$

Regarding the spectral function $N(x)$, assume the following:

I. The absolutely continuous component of the function $N(x)$ is different from zero.

$$\text{II.}\qquad a=\inf\left\{\sigma:\int_{\varepsilon}^{\infty}e^{-\sigma x}\,dN(x)<\infty\right\}<0\qquad(\varepsilon>0).$$

Under these assumptions $A(s)$ will be an analytic function in the half-plane $\operatorname{Re}s>a$. For real s, $A(s)$ is a convex downward function. Denote by z_0 the unique real minimum of the function $A(s)$, $z_0=A(s_0)$. If $\mathbf{M}\xi(1)<0$, then $s_0<0,\ z_0<0$; if $\mathbf{M}\xi(1)>0$, then $s_0>0,\ z_0<0$, and when $\mathbf{M}\xi(1)=0$ we have $s_0=z_0=0$.

On the ray $z>z_0$ the functions $\eta_+(z)$ and $\eta_-(z)$ are defined—the real roots of the equation $A(s)=z$. By the implicit-function theorem, $\eta_\pm(z)$ will be analytic in some neighborhood (in the z-plane) of the ray $z>z_0$. The only singular point of the functions $\eta_\pm(z)$ in this region is z_0, a branch point of the second order. Using condition I, one can prove the following assertions ([2,3]):

Lemma 1. For every $\sigma>a$ and $\tau_0>0$ there exists an $\varepsilon>0$ such that for $|\tau|>\tau_0$ the inequality

$$A(\sigma)>\operatorname{Re}A(\sigma+i\tau)+\varepsilon$$

holds.

Remark. From this lemma it follows, in particular, that the function $A(s)$ maps the line $\operatorname{Re}s=s_0$ into a certain contour $\mathcal{T}$ lying in the half-plane $\operatorname{Re}z\le z_0$.

Lemma 2. The function $\eta_+(z)$ is analytic in the domain D, situated to the right of the contour $\mathcal{J}$, and for $z\in D$ is the unique root of the equation $A(s)=z$ in the half-plane $\operatorname{Re}s\ge s_0$.

Lemma 3. For any sufficiently small $\delta>0$ there exists a $\gamma>0$ (common for all z on the line $\operatorname{Re}z=b,\ b\ge z_0$) such that, when $|\operatorname{Im}z|>\delta$, all roots of the equation $A(s)=z$ belonging to the half-plane $\operatorname{Re}s<s_0$ lie to the left of the vertical line $\operatorname{Re}s=\eta_-(\operatorname{Re}z)-\gamma$.

For the function $\varphi(z,s)$ the formula (5) holds:

$$\varphi(z,s)=\frac{\eta_+(x)-s}{s\eta_+(z-A(s))}.$$

The results of Lemmas 1-3 make it possible to pass from the double Laplace transform $\overline{\varphi}(z,s)$ to the asymptotics of the distribution function $F(t,x)$. A consequence of Lemmas 2, 3 and the residue theorem (6) is

Lemma 4. As $x\to\infty$ and $|\operatorname{Im}z|<\delta,\ \operatorname{Re}z>z_0-\delta$,

$$\varphi(z,x)=\frac{1}{z}-e^{\eta_-(z)x}\frac{\eta_+(z)-\eta_-(z)}{\eta_+(z)\eta_-(z)A'(\eta_-(z))}+\frac{1}{\eta_+(z)}O\big(e^{(\eta_-(\operatorname{Re}z)-\gamma)x}\big)\,.$$

If $|\operatorname{Im} z| > \delta,\ \operatorname{Re} z \geq z_0$, then

$$\varphi(z,x) = \frac{1}{z} + \frac{1}{\eta_+(z)} O\big(e^{(\eta_-(\operatorname{Re} z)-\gamma)x}\big)$$

(γ and δ are the constants appearing in Lemma 3).

The further asymptotic analysis is carried out by the saddle-point method or with the aid of a certain modification of this method ([4], Lemma 12). Following the arguments of A. A. Borovkov ([4]), we obtain theorems that are analogues of the corresponding theorems for sums of independent identically distributed random variables.

Theorem 1. Suppose $M\xi(1) \neq 0$, and x and t tend to infinity, with $x = o(t)$; then

$$\mathbf{P}\left\{\sup_{0\leq u\leq t} \xi(u) < x\right\} =$$

$$= F(x) + \frac{\omega_1^+ x}{2\sqrt{\pi c s_0^2} t^{3/2}} e^{s_0 x - z_0 t + H(\tau)t} \left(1 + \frac{1}{t} E_{1,2}\left(\frac{1}{t}, \tau\right)\right) + O\big(e^{(s_0-\gamma)x + z_0 t}\big);$$

$$F(x) = 1 - \frac{M\xi(1)}{A'(R)} e^{Rx} + O\big(e^{(R-\gamma)x}\big) \quad \text{when } M\xi(1) < 0.$$

If $M\xi(1) > 0$, then $F(x) \equiv 0$.

Here $\tau = x/t;\ R = \eta_-(0);\ H(\tau)$ is a function analytic at the point $\tau = 0$:

$$H(\tau) = -\frac{(\omega_1^-)^2}{4}\tau^2 + \frac{(\omega_1^-)^2\omega_2^-}{4}\tau^3 + \cdots;$$

$\omega_k^\pm$ are the coefficients in the expansion of the functions $\omega_\pm(p) = \eta_\pm(z_0 + p^2)$ in powers of p at the point $p = 0$ ($p = (z - z_0)^{1/2}$, and by $(z - z_0)^{1/2}$ we mean the principal value of the root):

$$\omega_+(p) = \eta_+(z_0 + p^2) = s_0 + \omega_1^+ p + \omega_2^+ p^2 + \cdots,$$

$$\omega_-(p) = \eta_-(z_0 + p^2) = s_0 + \omega_1^- p + \omega_2^- p^2 + \cdots;$$

c is the coefficient of $(s - s_0)^2$ in the Taylor expansion of the function $A(s)$ in a neighborhood of s_0; $E_{1,2}(1/t, \tau)$ is an expansion in powers of $1/t$ and τ, containing no terms of order $(1/t)^m \tau^n$ where simultaneously $m < 1,\ n < 2$ (the coefficients of this expansion are known).

Theorem 2. Let $\mathbf{M}\xi(1) = 0$, $x = X\sqrt{t}$, and $X = o(t^{1/6})$. Then, as $t \to \infty$,

$$\mathbf{P}\left\{\sup_{0\leqslant u\leqslant t} \xi(u) < X\sqrt{t}\right\} = 1 - \sqrt{\frac{2}{\pi}} \int_X^\infty e^{-u^2/2}\, du - e^{-\frac{1}{2}X^2} \sum_{j=1}^{\infty} \frac{1}{t^{j/2}} \Pi_{3j-1}(X) + O(e^{-\gamma x}),$$

where $\Pi_{3j-1}(X)$ are polynomials of degree $3j-1$ with known coefficients. (For simplicity we assume that $\mathbf{D}\xi(1) = 1$.)

Theorem 3. Let

$$\lim_{t\to\infty} \frac{x}{t} = \alpha > 0$$

and $\alpha = A'(\eta_-(0))$; then, as $t \to \infty$,

$$\mathbf{P}\left\{\sup_{0\leqslant u\leqslant t} \xi(u) < x\right\} = 1-(1-F(x))\Delta(z_\alpha) - \frac{\eta_+(z_\alpha) - \eta_-(z_\alpha)}{\sqrt{2\pi\alpha\eta_-''(z_\alpha)\eta_+(z_\alpha)\eta_-(z_\alpha)A'(\eta_-(z_\alpha))}} e^{tz_\alpha + xs_\alpha + tH_\alpha(\varepsilon)} \left(1 + \Xi_{1,1}\left(\frac{1}{t},\right.\right.$$

Here z_α is the real point at which the unique minimum of the function $z+\alpha\eta_-(z)$ is attained ($z_0 < z_\alpha < \infty$); $s_\alpha = \eta_-(z_\alpha)$; $\varepsilon = x/t - \alpha$; $\tau = x/t$; $\tau = \alpha + \varepsilon$; $H_\alpha(\varepsilon)$ is a function analytic at the point $\varepsilon = 0$:

$$H_\alpha(\varepsilon) = h_1\varepsilon + h_2\varepsilon^2 + \cdots;$$

the h_k depend on the derivatives of the function $\eta_-(z)$ at the point z_α and on the coefficients c_k in the expansion

$$z_\tau = z_\alpha + \sum_{k=1}^{\infty} c_k \varepsilon^k$$

(the coefficients c_k are found from the identity $1 + \tau\eta_-'(z_\tau) \equiv 0$); the value of $F(x)$ is given in Theorem 1; $\Xi_{1,1}(1/t, \varepsilon)$ is an expansion in powers of $1/t$ and ε with known coefficients, and this expansion contains no terms of order $(1/t)^m\varepsilon^n$, where simultaneously $m < 1$, $n < 1$;

$$\Delta(t) = \begin{cases} 1, & t < 0, \\ 0, & t > 0. \end{cases}$$

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Kyiv State University
named after T. G. Shevchenko

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Note: Figure translations are in progress. See original paper for figures.