CYBERNETICS AND CONTROL THEORY

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Abstract

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ON A LINEAR MODEL OF EXCHANGE

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In the work [1], Schwartz considers the following model of exchange. There are n producers and n commodities. A one-to-one correspondence is established between producers and commodities, and each commodity is ultimately used for the production of every other commodity (connectedness). For each producer, initial levels of production and inventory are specified, the components of which form vectors proportional to the proper vector of the economic matrix (the consumption-production matrix [1], p. 8), and the coefficient of increase of turnover c is defined (the ratio of the turnover of day t to the turnover of day $t-1$), which remains constant for the given model.

If the coefficient of growth of turnover is assumed to be the same in all branches of the economy, one can write the recurrence relations

$$\begin{aligned} a(t) &= \min(\{\gamma a(t-1) - b(t-1)\}^{+}, \quad b(t)/\gamma), \\ b(t) &= b(t-1) + \varepsilon a(t-1); \end{aligned} \tag{1}$$

$a(t)$ is the level of production on day t; $b(t)$ is the level of inventory on day t; $\bar{\gamma} = (c+2)\gamma - 1$; $\varepsilon = 1 - \gamma$; $c \geq 1$; $0 < \gamma < 1$ is the proper number of the economic matrix.

Relation (1) determines the dynamics of the economy, whose phase space is the domain $\{a \geq 0,\ b \geq \gamma a\}$ of the Euclidean plane (a, b) (in accordance with the notation of (1)), which we shall call admissible. The ray $b = \gamma a,\ a \geq 0$ is called the Scarcity line, hereafter Sc.l.; the ray $a = 0,\ b \geq 0$ is the Production cutoff, hereafter Pr.c.

Relations (1) can be written in vector form: $[a(t), b(t)] = \widetilde{\Lambda}[a(t-1), b(t-1)]$. If the point $[a(t), b(t)]$ is not a boundary point, then (1) reduces to the linear transformation

$$\begin{aligned} a(t) &= \bar{\gamma} a(t-1) - b(t-1), \\ b(t) &= \varepsilon a(t-1) + b(t-1), \end{aligned} \tag{2}$$

or, in vector form, $[a(t), b(t)] = \Lambda[a(t-1), b(t-1)]$.

The relation between $\widetilde{\Lambda}$ and Λ is as follows:

$$\widetilde{\Lambda}[a(t-1), b(t-1)] = [0, b(t)], \qquad \text{if } \tilde{a}(t) < 0,$$

$$[\tilde{a}, \tilde{b}] = \Lambda[a(t-1), b(t-1)],$$

$$\widetilde{\Lambda}[a(t-1), b(t-1)] = [b(t)/\gamma, b(t)], \qquad \text{if } \tilde{a}(t) > b(t)/\gamma,$$

$$\widetilde{\Lambda}[a(t-1), b(t-1)] = \Lambda[a(t-1), b(t-1)]$$

provided that $[\tilde{a}, \tilde{b}]$ is an admissible value of Λ.

The question posed in [1] is the following: by studying the sequence of iterations of Λ, determine the expansive or depressive nature of the model described above, which is governed by the relations between c and γ. In [1] this question is studied under particular assumptions concerning c and γ; in the present work a complete solution of the question is proposed.

Let us form the characteristic equation

$$\det(\Lambda - \lambda I) = 0, \qquad \lambda^2 - (\bar{\gamma} + 1)\lambda + (\bar{\gamma} + \varepsilon) = 0; \tag{3}$$

from equation (3) we derive:

$$\lambda_1 = \frac{\bar{\gamma}+1}{2} + \frac{\bar{\gamma}-1}{2}\left\{1 - \frac{4\varepsilon}{(\bar{\gamma}-1)^2}\right\}^{1/2},$$

$$\lambda_2 = \frac{\bar{\gamma}+1}{2} - \frac{\bar{\gamma}-1}{2}\left\{1 - \frac{4\varepsilon}{(\bar{\gamma}-1)^2}\right\}^{1/2}. \tag{4}$$

We shall distinguish three cases: I—the roots are real and distinct; II—the roots coincide; III—the roots are complex conjugates.

I. The equation $\Lambda\bar{w} = \lambda\bar{w}$ has nontrivial solutions: $\bar{w}_1(1, \bar{\gamma} - \lambda_1)$; $\bar{w}_2(1, \bar{\gamma} - \lambda_2)$ —eigenvectors of Λ in the (a, b)-system. Since $4\varepsilon < (\bar{\gamma} - 1)^2$, we have $\bar{\gamma} > 1$. Hence, $\bar{\gamma} > \lambda_1 > \lambda_2 > 1$, and $\bar{w}_2$ is situated above $\bar{w}_1$. In (1) it was shown that the expansive and depressive cases differ depending on the position of the vector $\bar{w}_2$ relative to Sc.l.: if $\bar{w}_2$ is situated above Sc.l.—expansive, below Sc.l.—depressive. Let us write down the conditions for both cases.

1. **Expansive case.**

a) $c \geq \frac{1}{\gamma^2} + \frac{1}{\gamma} - 1$ under the condition $\gamma \neq \frac{\sqrt{5}-1}{2}$;

b) $c > \sqrt{5} + 1$ under the condition $\gamma = (\sqrt{5}-1)/2$;

c) $\frac{2}{\gamma}\sqrt{1-\gamma}\,(1 + \sqrt{1-\gamma}) < c < \frac{1}{\gamma^2} + \frac{1}{\gamma} - 1$ under the condition $0 < \gamma < \frac{\sqrt{5}-1}{2}$.

2. **Depressive case.**

a) $\frac{2}{\gamma}\sqrt{1-\gamma}\,(1+\sqrt{1-\gamma}) < c < \frac{1}{\gamma^2}+\frac{1}{\gamma}$ under the condition $\frac{\sqrt{5}-1}{2} < \gamma \leq \frac{8}{9}$;

b) $1 \leq c < \frac{1}{\gamma^2} + \frac{1}{\gamma} - 1$ under the condition $\frac{8}{9} < \gamma < 1$.

It should be noted that the inequality

$$\frac{2}{\gamma}\sqrt{1-\gamma}\,(1 + \sqrt{1-\gamma}) \leq \frac{1}{\gamma^2} + \frac{1}{\gamma} - 1$$

always holds, and equality is possible only when $\gamma = \frac{\sqrt{5}-1}{2}$.

The trajectories of motion of the model are sequences of points lying on curves that constitute a family of parabolas with a common point at the origin of the (a, b)-system; $\bar{w}_2$ is the common tangent; $\bar{w}_1$ is an asymptotic direction (in the theory of differential equations such a family is called a node ([2])).

The geometric effect consists in the fact that the transformation pushes the points along the curves and brings them either onto Sc.l. or onto Pr.c. A distinctive feature compared with (1) is case 1b). In this case we either do not arrive at Sc.l. or are forced to leave it. The vector $\bar{w}_1$ is situated above Sc.l.; the expansive nature of the model does not ensure complete utilization of the stock.

II. The case of multiple roots presents additional mathematical peculiarities compared with I. On the basis of (4) we have

$$\lambda_1 = \lambda_2 = \lambda = 1 + \sqrt{1-\gamma}, \qquad c = \frac{2}{\gamma}(1 + \sqrt{1-\gamma}) - 2.$$

With the aid of a certain nonsingular transformation T, the matrix Λ is reduced to Jordan form ([3])

$$T^{-1}\Lambda T = \begin{Vmatrix} \lambda & 1 \\ 0 & \lambda \end{Vmatrix}, \qquad T = \begin{Vmatrix} \frac{1}{\sqrt{1-\gamma}} & \frac{1}{\sqrt{1-\gamma}-1} \end{Vmatrix}, \qquad T^{-1} = \begin{Vmatrix} 1 - \sqrt{1-\gamma} & 1 \\ \sqrt{1-\gamma} & -1 \end{Vmatrix}.$$

Let $\bar{a}_0\{a_0, b_0\}$ be some fixed vector of the (a, b)-system and $\bar{a}\{a, b\} = \Lambda^j \bar{a}_0$. Further, since

$$T^{-1}\Lambda^{jT} = (T^{-1}\Lambda T)^j = \begin{Vmatrix} \lambda^j & j\lambda^{j-1} \\ 0 & \lambda^j \end{Vmatrix},$$

where j is a positive integer, we may write

$$a' = \lambda^j a_0' + j\lambda^{j-1} b_0', \qquad b' = b_0' \lambda^j, \qquad \bar{a}'\{a', b'\} = T^{-1}\bar{a},$$

$$\bar{a}_0'\{a_0', b_0'\} = T^{-1}\bar{a}_0,$$

whence, putting

$$a' = y, \qquad b' = x, \qquad \frac{1}{\lambda \ln \lambda} = \mu, \qquad \frac{a_0'}{b_0'} - \frac{\ln |b_0'|}{\lambda \ln \lambda} = t,$$

we obtain

$$y = tx + \mu x \ln(x), \qquad -\infty < t < \infty, \tag{5}$$

i.e., a family of curves depending on the parameter t, in the (x, y)-system. The points $x_1 = -e^{-t/\mu}$, $x_2 = 0$, $x_3 = e^{-t/\mu}$ are the roots of the function y; the points $x_4 = -e^{-(\mu+t)/\mu}$, $x_5 = e^{-(\mu+t)/\mu}$ are, respectively, the points of maximum and minimum of the function y. The point $x_2 = 0$ is an inflection point of the function y, $y'(0) = \infty$. Further, $y'' > 0$ if $x > 0$, and $y'' < 0$ if $x < 0$ ($\mu > 0$, since $\lambda > 1$). The curves of the family (5), as in case I, form the angles $\bar{w}_1(1, \sqrt{1-\gamma})$, $\bar{w}_2(1, \sqrt{1-\gamma} - 1)$—the proper and associated vectors of the matrix Λ of the (a, b)-system. In passing from the (x, y)-system to the (a, b)-system, the general configuration of the curves is preserved. In this case the family is tangent to $\bar{w}_1$, passing from one side to the other, and intersects $\bar{w}_2$, which is always located outside the admissible domain. The expansive or depressive nature of the model is determined by the position of $\bar{w}_1$ relative to Sc.l.:

1. **Expansive case.**

$$0 < \gamma \leq \frac{\sqrt{5} - 1}{2}.$$

2. **Depressive case.**

$$\frac{\sqrt{5}-1}{2} < \gamma \le \frac{8}{9}.$$

As in I, 1 b), we cannot be on Sc.l., except for

$$\gamma = \frac{\sqrt{5}-1}{2}.$$

III. We turn to consideration of the last case. Let, as in II, $\bar{a}_0\{a_0, b_0\}$ be some fixed vector of the (a, b)-system and $\bar{a}\{a, b\} = \Lambda^j \bar{a}_0$ (j an integer). We have

$$T^{-1}\bar{a} = (T^{-1}\Lambda^{jT})T^{-1}\bar{a}_0, \qquad (T^{-1}\Lambda^{jT}) = (T^{-1}\Lambda T)^j. \tag{6}$$

The matrices T, T^{-1} have the form

$$T = \left\| \begin{matrix} 1 & 1 \\ \bar{\gamma}-\lambda_1 & \bar{\gamma}-\lambda_2 \end{matrix} \right\|, \qquad T^{-1} = \left\| \begin{matrix} \dfrac{\bar{\gamma}-\lambda_2}{\lambda_1-\lambda_2} & -\dfrac{1}{\lambda_1-\lambda_2} \\ \dfrac{\lambda_1-\bar{\gamma}}{\lambda_1-\lambda_2} & -\dfrac{1}{\lambda_1-\lambda_2} \end{matrix} \right\|.$$

On the basis of (6) we obtain

$$\lambda_1^j = \frac{a(\bar{\gamma}-\lambda_2)-b}{a_0(\bar{\gamma}-\lambda_2)-b_0}, \qquad \lambda_2^j = \frac{a(\lambda_1-\bar{\gamma})+b}{a_0(\lambda_1-\bar{\gamma})+b_0},$$

whence

$$\lambda_1^j = \frac{u+iv}{u_0+iv_0}, \qquad \lambda_2^j = \frac{u-iv}{u_0-iv_0}, \tag{7}$$

where $u = a\bar{\gamma} - b - \alpha a$, $u_0 = a_0\bar{\gamma} - b_0 - \alpha a_0$, $\lambda_1 = \alpha + i\beta$, $v = \beta a$, $v_0 = \beta a_0$, $\lambda_2 = \alpha - i\beta$.

From relation (7) it follows that

$$|\lambda_1|^{2j}\frac{u^2+v^2}{u_0^2+v_0^2}, \qquad \arg(\lambda_1^j) = \operatorname{arctg}\frac{u}{v} - \operatorname{arctg}\frac{u_0}{v_0} \pm k\pi,$$

which together determine

$$\frac{1}{2\ln\lambda_1}\ln\frac{u^2+v^2}{u_0^2+v_0^2} = \frac{1}{\arg\lambda_1}\left\{\operatorname{arctg}\frac{u}{v} - \operatorname{arctg}\frac{u_0}{v_0} \pm k\pi\right\}, \tag{8}$$

i.e., a family of curves of the (u, v)-system depending on a certain parameter.

Equation (8) becomes more transparent if one passes to the polar coordinate system:

$$\rho = C \exp\left[\frac{\ln|\lambda_1|}{\arg\lambda_1}\,\varphi\right], \qquad \rho^2 = u^2 + v^2, \qquad \varphi = \operatorname{arctg}\frac{u}{v},$$

$$C = \sqrt{u_0^2 + v_0^2}\exp\left[\frac{\ln|\lambda_1|}{\arg\lambda_1}\left\{\pm k\pi - \operatorname{arctg}\frac{u_0}{v_0}\right\}\right]. \tag{9}$$

From the point of view of geometric classification, we shall distinguish the following three cases:

1. $|\lambda_1|^2 = (c+1)\gamma > 1$ –a family of spirals that emerge from the origin of the (u, v)-system, unwinding to infinity (focus).
2. $|\lambda_1|^2 = (c+1)\gamma = 1$ –a family of circles with common center at the origin (center).
3. $|\lambda_1|^2 = (c+1)\gamma < 1$ –a family of spirals that emerge from an infinitely distant point, winding onto the origin of the (u, v)-system (focus).

When passing from the (u, v)-system to the (a, b)-system, the general configuration of the family is preserved. It should be noted, however, that with a change in the orientation of the coordinate system, the orientation of the motion of the economy changes. In all three cases the nature of the model is depressive.

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REFERENCES

1. J. Schwartz, *Lectures on the Mathematical Method in Analytical Economics*, 1, N. Y., 1961.

2. V. V. Stepanov, *Course of Differential Equations*, Moscow, 1959.

3. I. M. Gel' fand, *Lectures on Linear Algebra*, 2nd ed., Moscow–Leningrad, 1951.

Note: Figure translations are in progress. See original paper for figures.