A GENERAL THEORY OF VARIATIONAL PROBLEMS WITH APPLICATIONS TO OPTIMAL CONTROL

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Abstract

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A GENERAL THEORY OF VARIATIONAL PROBLEMS WITH APPLICATIONS TO OPTIMAL CONTROL

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In the present note a very general variational problem is formulated and the corresponding necessary condition for extremality is given, containing as special cases the known first-order necessary conditions of the calculus of variations, as well as Pontryagin' s maximum principle and its various generalizations.

Let B and Q be two sets in a locally convex linear topological space J. It is assumed that $0 \in B \cap Q$ and that the topology in J induces the usual Euclidean topology on every finite-dimensional subspace of J. Let $\hat{\varphi}$ be a continuous mapping of some neighborhood N of zero in J into R^m (m-dimensional Euclidean space), with $\hat{\varphi}(0) = 0$, and let $Y = \{x : x \in N,\ \hat{\varphi}(x) = 0\}$. We shall say that zero in J is a $(Q, B, \hat{\varphi})$-extremal if there exists a neighborhood N^* of zero in J such that $N^* \cap B \cap Q \cap Y = \{0\}$.

To obtain meaningful necessary conditions for extremality, it is necessary to assume that B, Q, and $\hat{\varphi}$ possess certain properties which, in a known sense, may be characterized as the existence of "convex first-order approximations." Precisely, these properties may be formulated as follows. It is assumed that there exists a continuous linear mapping $\hat{l}$ from J into R^m such that

$$\frac{\hat{\varphi}(\varepsilon y)}{\varepsilon} \longrightarrow \hat{l}(x) \quad \text{as } y \to x,\ \varepsilon \to 0 \tag{1}$$

for every $x \in J$.

It is also assumed that there exists a convex cone $Z \subset J$ with vertex at 0, possessing the following property: for each ray $\rho \subset Z$ there exist a cone Z_ρ and a neighborhood N_ρ of zero in J such that: a) the vertex of Z_ρ is at 0; b) $Z_\rho \subset Z$; c) Z_ρ has a nonempty open core and ρ is an interior ray of Z_ρ; d) $Z_\rho \cap N_\rho \subset B$. In this case we say that Z is an *inner cone* for B at 0.

Next we shall assume the existence of a convex subset K of J, containing 0 and possessing the following property: if S is an arbitrary simplex from K with vertices $x_1, \dots, x_\nu$, and $\widetilde{N}$ is an arbitrary neighborhood of zero in J, then there exists a number $\varepsilon_0 > 0$ such that for every ε, $0 < \varepsilon < \varepsilon_0$, there is a continuous mapping ζ_ε from $P^\nu = \{\beta = (\beta_1, \dots, \beta_\nu) : \beta_i \geq 0,\ i = 1, \dots, \nu,\ \sum_{i=1}^{\nu} \beta_i = 1\}$ into J (this mapping may depend both on ε and on S and $\widetilde{N}$), satisfying the condition

$$\zeta_\varepsilon(\beta) \in \left\{ \varepsilon \left(\sum_{i=1}^{\nu} \beta_i x_i + \widetilde{N} \right) \right\} \cap Q \quad \text{for every } \beta \in P^\nu.$$

In this case we shall say that K is a *convex first-order approximation* for Q at zero.

Let $\Pi = \{x : x \in J,\ \hat{l}(x) = 0\},\ Z' = Z \cap \Pi$. Our last assumption is that $Z' \neq \Pi,\ Z' \neq \{0\}$.

As is indicated below, most optimal-control problems can be reduced to the scheme described.

Necessary conditions for optimality are given by the following two theorems.

Theorem 1. If 0 is a $(Q, B, \hat{\varphi})$-extremal and Q, B, and $\hat{\varphi}$ satisfy the conditions listed above, then K and Z' are separable, i.e., there exists a linear continuous nonzero functional l^* on J such that

$$l^*(x) \leq 0 \leq l^*(y)$$

for all $x \in K,\ y \in Z'$.

Theorem 2. If l^* is a linear functional on J such that $l^*(y) \geq 0$ for $y \in Z'$, then

$$l^*(x) = \bar{l}(x) + a \cdot \hat{l}(x)$$

for any $x \in J$, where a is a vector in R^m and $\bar{l}(y) \geq 0$ for any $y \in Z$.

Let φ_0 be a real function defined in some neighborhood N_0 of zero in J, and suppose that there exists a continuous convex functional l_0 from J to R^1 such that $l_0(x) < 0$ for some $x \in J$ and

$$\varphi_0(\varepsilon y)/\varepsilon \underset{\substack{y \to x \\ \varepsilon \to 0+}}{\longrightarrow} l_0(x) \quad \text{for any } x \in J. \tag{2}$$

If

$$B = \{x : x \in N_0,\ \varphi_0(x) < 0\} \cup \{0\}$$

and

$$Z = \{x : x \in J,\ l_0(x) < 0\} \cup \{0\},$$

then Z is an interior cone for B at 0.

We now indicate how the formulated results are to be applied to optimal-control problems. Let J_0 be the space of continuous mappings of the compact interval

I into R^n, with the usual metric of uniform convergence. Let $J = J_0 \times A$, where A is a convex set in R^k, and U an arbitrary set in R^r. Denote by Ω the set of all essentially bounded measurable functions from I into U. Let $f(x,u,t)$ be a function from $R^n \times U \times I$ into R^n, of class C^1 with respect to x and measurable with respect to (u,t), and suppose that $f(x,u,t)$, $f_x(x,u,t)$ are bounded on $X \times I$ for any function $u \in \Omega$ and compact set $X \subset R^n$.

Further, let Q_0 be the set of all absolutely continuous functions $x(t) \in J_0$ satisfying the equation

$$\dot{x}(t) = f(x(t), u(t), t) \tag{3}$$

for a suitable choice of $u(t) \in \Omega$. Let $\tilde{\varphi}$ be a continuous function from an open set $W \subset J$ into R^m, and let $\varphi_0, \varphi_1, \dots, \varphi_\mu$ be functionals defined on W (not necessarily linear). Consider the following optimal problem: find an element $z \in W \cap (Q_0 \times A)$ satisfying the conditions $\tilde{\varphi}(z) = 0,\ \varphi_i(z) \le 0,\ i = 1, \dots, \mu$, and minimizing the functional $\varphi_0(z)$. Let $z^* = (x^*, \alpha^*)$, $x^* \in Q_0,\ \alpha^* \in A$, be a solution of the problem, and put

$$\varphi(z) = \tilde{\varphi}(z + z^*), \quad \varphi_i = \varphi_i(z + z^*), \quad \varphi_0(z) = \varphi_0(z + z^*) - \varphi_0(z^*).$$

Assume that conditions (1) and (2) are satisfied for the functions $\hat{\varphi}, \varphi_i,\ i = 0, 1, \dots, \mu$, where l is a linear continuous mapping onto all of R^m, and the functionals l_i are convex and continuous. Discarding, if necessary, some of the φ_i, assume that $\varphi_i(0) = 0$. If we put

$$Q = (Q_0 - x^*) \times (A - \alpha^*), \quad B = \{z : z \in W,\ \varphi_i(z) < 0,\ i = 0, 1, \dots, \mu\} \cup \{0\},$$

then we obtain that 0 is a $(Q, B, \hat{\varphi})$-extremal and that the set

$$Z = \{x : x \in J,\ l_i(x) < 0,\ i = 0, 1, \dots, \mu\} \cup \{0\}$$

is an interior cone for B at 0. Using the main result of [2], it is easy to show that Q admits at zero a convex first-order approximation.

If the "differentials" $l_0, l_1, \dots, l_\mu$ are linear and $\bar{l}$ is a linear functional on J_μ such that $\bar{l}(y) \ge 0$ for any $y \in Z$, then

$$\bar{l} = \sum_{i=0}^{\mu} \alpha_i l_i, \qquad \alpha_i \le 0.$$

Using Theorems 1 and 2, we conclude that in the case under consideration

there exist real numbers $\alpha_i,\ i = 0, 1, \dots, \mu$, and a vector $\hat{a} \in R^m$ (not all equal to zero), such that

$$\hat{a} \cdot l(x) + \sum_{i=0}^{\mu} \alpha_i l_i(x) \le 0, \qquad x \in K, \qquad \alpha_i \le 0, \qquad i = 0, 1, \dots, \mu. \tag{4}$$

It is easy to show that (4) contains Pontryagin' s maximum principle as a special case.

If $\mu > 0$, $\varphi_i = \sup_{t\in I} g(x(t))$, $i = 1, \dots, \mu$, where g is a function of class C^1 from R^n to R^1, then the continuous convex functionals

$$l_i = \sup_{t\in T}[\operatorname{grad} g(x^*(t))] \cdot x(t), \qquad T = \{t : t \in I,\ g(x(t)) = 0\}, \tag{5}$$

$i = 1, \dots, \mu$, satisfy condition (2), and Theorems 1 and 2 give necessary conditions for optimal problems with constrained phase coordinates, analogous to the conditions in ([1]), Chap. VI and ([3,4]). It can also be shown that formula (4) is valid in this case as well and gives new necessary conditions for optimality. If (5) is also valid for $i = 0$, we obtain necessary conditions for minimax problems. Similar considerations make it possible to obtain necessary conditions for discrete optimal processes ([5]) and for optimal processes with impulsive control.

In conclusion, let us note that optimal processes from a similar point of view were recently considered by A. Ya. Dubovitskii and A. A. Milyutin ([7]), and our results in many respects supplement the results contained in ([7]).

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Note: Figure translations are in progress. See original paper for figures.