THE NATURE OF THE SINGULARITY IN THE GRAVITATIONAL COLLAPSE OF A STAR

Astronomy

1966

SovietRxiv

Abstract

Full Text

UDC 523.12

Astronomy

M. A. Podurets

THE NATURE OF THE SINGULARITY IN THE GRAVITATIONAL COLLAPSE OF A STAR

(Presented by Academician Ya. B. Zel̆dovich on 20 VII 1965)

In many cases, integration in time of partial differential equations leads to singularities in the solution. This makes direct numerical integration impossible and forces one to investigate the analytic nature of the solution near the singularity; moreover, it proves possible to construct a limiting solution that is practically independent of the initial conditions. A well-known self-similar solution for a converging hydrodynamic shock wave may serve as an example. An analogous situation also occurs in the case of the gravitational collapse of a star.

Let us consider a collapsing spherically symmetric star. In a comoving reference system, each particle passes inside its own gravitational radius, while the energy density and pressure along each trajectory increase without bound. Owing to the influence of the pressure gradient, collapse (the divergence of the density to infinity) occurs not simultaneously (not at one and the same instant of the comoving time τ) throughout the entire star. The density first becomes infinite at the center of the star, and subsequently at other points; the totality of these events in the (τ, R) plane gives the line $\tau = \tau_0(R)$ ([1]). This qualitatively distinguishes the collapse of a star from the collapse of a closed homogeneous universe, where the pressure gradient is absent and the collapse occurs simultaneously. Let us note that in a comoving reference system the concept of simultaneity is a quite definite concept, since the time τ is determined up to a transformation of it into itself, and by such a transformation one cannot convert $\tau = \tau_0(R)$ into $\tau = \text{const}$.

The interval in a spherical comoving reference system has the form

$$ds^2 = e^{\sigma} d\tau^2 - e^{\omega} dR^2 - r^2(\sin^2\theta \, d\varphi^2 + d\theta^2).$$

We write the field equations in the form ([2])

$$m = \tfrac{1}{2} r(1 + e^{-\sigma}\dot{r}^2 - e^{-\omega} r'^2), \tag{1}$$

$$m' = 4\pi\varepsilon r^2 r', \tag{2}$$

$$\dot{m} = -4\pi p r^2 \dot{r}, \tag{3}$$

$$\sigma' = -\frac{2p'}{p+\varepsilon}, \tag{4}$$

$$\dot{\omega} = -\frac{2\dot{\varepsilon}}{p+\varepsilon} - 4\frac{\dot{r}}{r}. \tag{5}$$

Here a dot denotes differentiation with respect to τ, a prime with respect to R; p is the pressure; ε is the energy density; m is the total mass inside the radius R, the mass that would be felt by a distant observer if the matter ended at this R. We are interested in the behavior of the solution near the line $r = 0$, where $p \to \infty$, $\varepsilon \to \infty$.

Naturally, here one may use a limiting equation of state—the equation of state of an ultrarelativistic gas, $p = \frac{1}{3}\varepsilon$, or the limiting stiff equation of state $p = \varepsilon$ ([3]).

Let us first use the equation $p = {}^1/_3\,\varepsilon$. In this case equations (4) and (5) can be integrated, and we obtain

$$e^{-\sigma} = \varphi_1(\tau)\sqrt{\varepsilon}, \qquad e^{-\omega} = \varphi_2(R)\varepsilon\sqrt{\varepsilon}\, r^4; \tag{6}$$

$\varphi_1(\tau)$ and $\varphi_2(R)$ are arbitrary functions, and, owing to the ambiguity in the definition of the variables τ and R, they may be set equal to unity. We note here that in a closed homogeneous universe ε is a function only of τ, and one may set $e^{\sigma} = 1$.

The system takes the form

$$m = {}^1/_2 r(1 + \sqrt{\varepsilon}\, r_\tau - \varepsilon\sqrt{\varepsilon}\, r^4 r'^2), \tag{7}$$

$$m' = 4\pi\varepsilon r^2 r', \tag{8}$$

$$\dot{m} = -{}^4/_3 \pi\varepsilon r^2 \dot{r}. \tag{9}$$

The usual way of investigating the nature of a singularity consists in assuming some analytic character of the solution (factoring out nonanalytic factors, self-similarity, etc.). However, it would be hopeless to try to do this in the variables τ and R, in view of their aforementioned ambiguity.

First of all, this applies to the variable τ; therefore we shall replace it by the variable r, i.e., pass to the independent variables r, R. In these variables the system of equations has the form

$$m = {}^1/_2 r + {}^1/_2 r\sqrt{\varepsilon}\,\frac{1}{(\partial\tau/\partial r)^2}\left[1 - r^4\varepsilon(\partial\tau/\partial R)^2\right], \tag{10}$$

$$\partial m/\partial r = -{}^4/_3\pi\varepsilon r^2, \tag{11}$$

$$\frac{\partial\tau}{\partial r}\frac{\partial m}{\partial R} = -{}^{16}/_3\pi\varepsilon r^2\frac{\partial\tau}{\partial R}. \tag{12}$$

Let us now make an assumption about the analytic properties of the function $\varepsilon(r, R)$. Namely, let us assume that its nonanalyticity is the same for all R (such an assumption is natural, since there are no distinguished values of R *), i.e., let us assume that ε can be represented in the form $\varepsilon = f(r)\varepsilon_1(r, R)$, where the function ε_1 is analytic, i.e.,

$$\varepsilon(r, R) = f(r)\,[a(R) + rb(R) + ...].$$

Since derivatives of ε do not enter the system of equations, we may discard decreasing terms and approximately write

$$\varepsilon = f(r)a(R). \tag{13}$$

Our problem consists in determining the function $f(r)$.

Representation (13) makes it possible to integrate equations (11) and (12) near $r = 0$.

From equation (11) we obtain

$$m = -{}^4/_3\pi a(R)\xi(r) + m_0(R), \tag{14}$$

where the function $\xi(r)$ satisfies the equation

$$d\xi/dr = f(r)r^2. \tag{15}$$

In equation (12), near $r = 0$, one may set $\partial\tau/\partial r = d\tau_0(R)/dr$, and this quantity is finite and nonzero by virtue of the basic assumption of non-simultaneity of the collapse.

* The only exception may be a single point—the center of the star, i.e., the particle at which $r \equiv 0$. We, however, are interested in the tending to zero of the functions $r(\tau, R)$ for all values of R.

Having done this, let us express from (12) the quantity $\partial\tau/\partial r$

$$\frac{\partial\tau}{\partial r} = -\frac{16}{3}\pi a(R)\frac{d\tau_0}{dR}\frac{f(r)r^2}{(dm_0/dR) - \frac{4}{3}\pi\xi(r)(da/dR)}. \tag{16}$$

Let us now consider the order of the terms on the right-hand side of (10). The density becomes infinite after each particle passes through its gravitational radius; the totality of these events gives the line $2m = r$. After this, right up to the collapse, $m > \frac{1}{2}r$, since m is a nondecreasing function (see (3)). This means that the second term on the right-hand side of (10) is larger than the third. There may then be two possibilities: either the second is the leading term, or the order of the second and third terms is the same.

Let us first consider the second possibility. In view of the finiteness of $d\tau_0/dR$, we obtain $\varepsilon \sim r^{-4}$, i.e. $f(r) \sim r^{-4}$. From (14) and (15) we obtain the order of the quantities m and $\partial\tau/\partial r$: $m \sim r^{-1}$, $\partial\tau/\partial r \sim r^{-1}$. On the other hand, from (10) we obtain the order of magnitude of m:

$$m \sim r\sqrt{f}\,\frac{1}{(\partial\tau/\partial r)^2} \sim r,$$

which contradicts the estimate of the order of the mass from equation (14).

It follows that near $r = 0$ the principal term in (10) is the second; i.e., in the limit

$$m = \frac{1}{2}r\sqrt{\varepsilon}\,\frac{1}{(\partial\tau/\partial r)^2}. \tag{17}$$

Let us consider the behavior of the mass m as $r \to 0$. Its limit may be either finite, or the mass may grow without bound (as, for example, in the Friedmann solution for a homogeneous universe). We shall show that the limit of the mass is finite.

For the proof, suppose the contrary. This means that $\xi(r) \to -\infty$, and from (16) we have $\partial\tau/\partial r \sim -f(r)r^2/\xi(r)$. Then from (14) and (17) we obtain $f(r) \sim r^{-2}\xi^{2/3}$. Hence, with the help of (15), we obtain an equation for $\xi(r)$: $d\xi/dr = A(R)\xi^{2/3}$, whence $r = 3A\xi^{1/3} + B$. It then turns out that the limit of ξ, and consequently also of the mass, as $r \to 0$ is finite. The contradiction to the assumption made proves the finiteness of m as r tends to zero.

Using the proved boundedness of the mass, immediately from (14), (16), and (17) we obtain the desired dependence $f = r^{-2}$. Thus, in the variables τ, R near the line $\tau = \tau_0(R)$ (i.e., near $r = 0$) the limiting behavior of the solution is characterized by the following relations:

$$e^{\sigma} \sim r, \qquad e^{\omega} \sim r^{-1}, \qquad \varepsilon \sim r^{-2}, \qquad \dot{r} \sim r^{0}, \qquad \dot{m} \sim r^{0}.$$

Let us also give analogous relations for the collapse of a star in the case of the stiffest equation of state $p = \varepsilon$:

$$e^{\sigma} \sim r^{3}, \qquad e^{\omega} \sim r^{-1}, \qquad \varepsilon \sim r^{-3}, \qquad \dot{r} \sim r, \qquad \dot{m} \sim r^{0}.$$

Received
28 V 1965

CITED LITERATURE

1. M. A. Podurets, DAN, **154**, 300 (1964).

2. M. A. Podurets, Astr. Zhurn., **41**, 28 (1964).

3. Ya. B. Zel' dovich, ZhETF, **41**, 1609 (1961).

Note: Figure translations are in progress. See original paper for figures.