FUNCTIONALS OF DERIVATIVES ON THE COMPLEX PLANE

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Abstract

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FUNCTIONALS OF DERIVATIVES ON THE COMPLEX PLANE

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In this article the following problems are considered.

1. Among all algebraic polynomials of degree n with real coefficients satisfying the condition $\max_{[0,1]} |P_n(x)| = 1$, find the one which, at a given point $x = z_0$ of the complex plane, gives
$$\max |\operatorname{Re} P_n^{(k)}(x)|$$
or
$$\max |\operatorname{Im} P_n^{(k)}(x)|$$
($k = 1, 2, \dots, n$). The solution is given in the domain $|x| \geq \rho_{0,k}$, where $\rho_{0,k} < 1$; it is sufficient to consider $\operatorname{Im} x > 0$.
2. In the same class of polynomials, find a polynomial which at the given point $x = z_0$ gives
$$\max \left|\operatorname{Re} \frac{\partial^k P_n}{\partial \varphi^k}\right|$$
or
$$\max \left|\operatorname{Im} \frac{\partial^k P_n}{\partial \varphi^k}\right|$$
($k > 0$ an integer). The solution is given in the domain $|x| \geq \rho_{1,k}$, where $\rho_{1,k} < 1$.

We shall consider $\operatorname{Re} P_n^{(k)}(x)$ at the point

$$x = z_0 = \rho(\cos\varphi + i \sin\varphi)$$

as a linear functional $F^{(k)}_{z,\cos}$, defined on $\{P_n(x)\}$ by the finite sequence

$$0_0,\ 0_1, \dots,\ 0_{k-1},\ k!,\ (k+1)!\rho\cos\varphi, \dots, \frac{n!}{(n-k)!}\rho^{n-k}\cos(n-k)\varphi \qquad (1)$$

$$(\rho > 0;\ 0 \leq \varphi \leq \pi).$$

Under the same interpretation, $\operatorname{Im} P_n^{(k)}(x)$ at the point z_0 will be given by the functional $F_{z,\sin}^{(k)}$:

$$0_0,\ 0_1, \dots,\ 0_k,\ (k+1)!\rho\sin\varphi, \dots, \frac{n!}{(n-k)!}\rho^{n-k}\sin(n-k)\varphi \tag{2}$$

$$(\rho > 0;\ 0 \le \varphi \le \pi).$$

Similarly, $\operatorname{Re} \partial^k P_n/\partial\varphi^k$ will be considered as the functional $F_{\varphi,\cos}^{(k)}$, defined by the sequence

$$0,\ \rho\cos\varphi,\ 2^k\rho^2\cos 2\varphi, \dots,\ n^k\rho^n\cos n\varphi \qquad (k \text{ even}), \tag{3}$$

and as the functional $F_{\varphi,\sin}^{(k)}$, defined by the sequence

$$0,\ \rho\sin\varphi,\ 2^k\rho^2\sin 2\varphi, \dots,\ n^k\rho^n\sin n\varphi \qquad (k \text{ odd}). \tag{4}$$

We shall interpret $\operatorname{Im} \partial^k P_n/\partial\varphi^k$ as $F_{\varphi,\sin}^{(k)}$ for even k and as $F_{\varphi,\cos}^{(k)}$ for odd k on the same set of polynomials. Thus,

$$F_{z,\cos}^{(k)}[P_n(x)] = \operatorname{Re} P_n^{(k)}(z_0); \qquad F_{z,\sin}^{(k)}[P_n(x)] = \operatorname{Im} P_n^{(k)}(z_0),$$

and so on.

For each of the functionals written above, for any $x = z_0$ there exists a polynomial $Q_n(x, z_0)$, called extremal, such that, with

$$\max_{[0,1]} |Q_n(x, z_0)| = 1,$$

the norm of the corresponding functional is attained on it. For example,

$$N_{z,\cos}^{(k)}(z_0) = \operatorname{Re} Q_n^{(k)}(z_0, z_0).$$

Then for any polynomial $\mathcal{L}_n(x)$ with real coefficients,

$$\max \left|\operatorname{Re} \mathcal{L}_n^{(k)}(z_0)\right| \le M_{\mathcal{L}} \cdot N_{z,\cos}^{(k)}(z_0) = M_{\mathcal{L}} \cdot \operatorname{Re} Q_n^{(k)}(z_0, z_0),$$

where

$$M_{\mathcal{L}} = \max_{[0,1]} |\mathcal{L}_n(x)|.$$

Thus, the problem consists in finding the extremal polynomials of the functionals (1), (2), (3), and (4).

Consider two families of algebraic polynomials:

$$L_{n+1}(x) = L_{n+1,0}(x) = \prod_{i=0}^{n}(x - \sigma_i) = \sum_{m=0}^{n+1} a_m x^m,$$

where

$$0 \leqslant \sigma_0 < \sigma_1 < \cdots \sigma_n \leqslant 1, \quad L_{n+1,k+1}(x) = \frac{x}{n+1} L'_{n+1,k}(x); \tag{5}$$

$$\Lambda_{n,p}(x) = \frac{L_{n+1}(x)}{x-\sigma_p} = \sum_{m=0}^{n} a_m^{(p)} x^m; \qquad \Lambda_{n,k+1,p}(x) = \frac{x}{n} \Lambda'_{n,k,p}(x) \tag{6}$$

$$(k \geqslant 0 \text{ integer}; \; p = 0, 1, \dots, n).$$

Denote by $\{x_{k,m}^{(p)}\}_{m=1}^{m=n}$ the roots of the polynomial $\Lambda_{n,k,p}(x)$; by $\{x_{k,m}\}_{m=0}^{m=n}$ the roots of the polynomial $L_{n+1,k}(x)$.

Lemma 1. *All roots of the polynomials $\Lambda_{n,k,p}(x)$ and $L_{n+1,k}(x)$ are real and nonnegative; moreover:*

$$\begin{array}{lll} 1) \;\; x_{k,m} < x_{k-1,m} & (m = 1, 2, \dots, n); & x_{k,0} \leqslant x_{k-1,0}, \\ 2) \;\; x_{k,m}^{(p)} < x_{k,m}^{(p-1)} & (m = 2, 3, \dots, n); & x_{k,1}^{(p)} \leqslant x_{k,1}^{(p-1)}, \\ 3) \;\; x_{k,m-1} < x_{k,m}^{(p)} < x_{k,m} & (m = 2, 3, \dots, n); & x_{k,0} \leqslant x_{k,1}^{(p)} < x_{k,1}. \end{array}$$

We shall prove the inequalities. The first follows from the recurrence formula (5). Inequalities 2) and 3) follow from formulas (5) and (6) and the results of E. V. Voronovskaya (see ([1]), p. 370).

Introduce the trigonometric polynomials

$$\frac{1}{(n+1)^k} F_{\varphi,\cos}^{(k)}[L_{n+1}(x)] = \sum_{m=1}^{n+1} a_m \rho^m \left(\frac{m}{n+1}\right)^k \cos m\varphi \quad (k = 1, 2, \dots),$$

$$\frac{1}{n^k} F_{\varphi,\cos}^{(k)}\left[\frac{L_{n+1}(x)}{x-\sigma_p}\right] = \sum_{m=1}^{n} a_m^{(p)} \rho^m \left(\frac{m}{n}\right)^k \cos m\varphi \quad (k = 1, 2, \dots).$$

Denote the real roots φ of the polynomial $F_{\varphi,\cos}^{(k)}[L_{n+1}(x)]$, for $\rho = \text{const} \geqslant x_{k,n}$, by $\{\psi_{k,m}(\rho)\}_{m=0}^{m=n}$, and the roots of $F_{\varphi,\cos}^{(k)}[L_{n+1}(x)/(x-\sigma_p)]$ by $\{\psi_{k,m}^{(p)}\}_{m=1}^{m=n}$.

Lemma 2. *For $\rho \geqslant \rho_{1,k} = x_{n,k}$ ($x_{k,n} < 1$ and $x_{k,n} \to 0$ as $k \to \infty$), the following inequalities hold:*

$$\begin{array}{ll} 1) \;\; \psi_{k,m}^{(p)} < \psi_{k,m}^{(p+1)} & (m = 1, 2, \dots, n; \; p = 0, 1, \dots, n); \\ 2) \;\; \psi_{k,m-1} < \psi_{k,m}^{(p)} < \psi_{k,m} & (m = 1, 2, \dots, n; \; p = 0, 1, \dots, n). \end{array}$$

For the proof, note that

$$(1/(n+1)^k)F^{(k)}_{\varphi,\cos}[L_{n+1}(x)] = F_{\cos}[L_{n+1,k}(x)],$$

$$(1/n^k)F^{(k)s}_{\varphi,\cos}[\Lambda_{n,p}(x)] = F_{\cos}[\Lambda_{n,k,p}(x)],$$

where the functional $F_{\cos}$ on $\{P_n\}$ is defined by the finite sequence $1, \rho\cos\varphi, \rho^2\cos 2\varphi, \dots, \rho^n \cos n\varphi$, and the inequalities follow from Lemma 1 and Theorem 1 of note [3].

Denote: $(\tau_m)_0^n$ are the nodes of the polynomial $T_n(x) = \cos n \arccos(2x-1)$,

$$R_{n+1}(x) = \prod_{i=0}^{n}(x-\tau_i);$$

$\rho_{0,k}$ is the largest root of the polynomial

$$\frac{d^k}{dx^k}(R_{n+1}(x)/x).$$

Theorem 1. *In the semicircle of radius $\rho \geqslant \rho_{0,k}$ there are $n-k+1$ arcs $[\alpha^{(k)}_m(\rho); \beta^{(k)}_m(\rho)]$ $(k = 1, 2, \dots, n;\ m = 1, 2, \dots, n-k+1)$, at whose points, for the functional (1), one of the polynomials $\pm T_n(x)$ is extremal; the arc boundaries are $\alpha^{(k)}_1 = 0;\ \alpha^{(k)}_2, \alpha^{(k)}_3, \dots, \alpha^{(k)}_{n-k+1}$ are the roots of the equation $F^{(k)}_{z,\cos}[R_{n+1}(x)/(x-1)] = 0;\ \beta^{(k)}_1, \beta^{(k)}_2, \dots, \beta^{(k)}_{n-k}$ are the roots of the equation $F^{(k)}_{z,\cos}[R_{n+1}(x)/x] = 0$, and $\beta^{(k)}_{n-k+1} = \pi$. We shall call these arcs Chebyshev arcs.*

Proof. Apply the extremality criterion, containing two conditions (see [1,2]), to $P_n(x) = \pm T_n(x)$. In this case $s = n+1$ and condition 1) drops out, while condition 2) consists in the fact that $\delta_0, \delta_1, \dots, \delta_n$ have alternating signs. For the segment (1) the system of equations has the form

$$\sum_{p=0}^{n} \delta_p \tau_p^m = 0 \quad (m = 0, 1, \dots, k-1),$$

$$\sum_{p=0}^{n} \delta_p \tau_p^m = \frac{m!}{(m-k)!}\rho^{m-k}\cos(m-k)\varphi \quad (m = k, k+1, \dots, n),$$

and its solution is given by the formula

$$\delta_p = \frac{(-1)^{n-p}}{\prod_{p\neq m}|\tau_p - \tau_m|} F^{(k)}_{z,\cos}\left[\frac{R_{n+1}(x)}{x-\tau_p}\right] \quad (p = 0, 1, \dots, n).$$

Let us note that

$$F^{(k)}_{z,\cos}\left[\frac{R_{n+1}(x)}{x-\tau_p}\right] = F_{\cos}\frac{d^k}{dx^k}\left(\frac{R_{n+1}(x)}{x-\tau_p}\right); \quad F^{(k)}_{z,\cos}[R_{n+1}(x)] = F_{\cos}[R^{(k)}_{n+1}(x)].$$

Denote by $\Phi_j(x) = R_{n+1}(x)/(x-\tau_j)$ $(j = 0, 1, \ldots)$; by $\{\xi^{(m)}_{j,k}\}^{n-k}_{m=1}$ the roots of $\Phi^{(k)}_j(x)$; by $\{\theta^{(k)}_m\}^{n-k+1}_{m=1}$ the roots of the polynomial $R^{(k)}_{n+1}(x)$; by $\{\gamma^m_{j,k}(\rho)\}^{n-k}_{m=1}$ the roots of $F_{\cos}[\Phi^{(k)}_j(x)]$ on $[0,\pi]$ $(\rho \geq \rho_{0,k})$; and by $\{\xi^{(k)}_m(\rho)\}^{n-k+1}_{m=1}$ the roots of $F_{\cos}[R^{(k)}_{n+1}(x)]$ on $[0,\pi]$.

For the roots of the polynomials $\{\Phi^{(k)}_j(x)\}^n_{j=0}$ we have [2]:

$$\xi^{(m)}_{0,k} > \xi^{(m)}_{1,k} > \cdots > \xi^{(m)}_{n,k} \quad (m = 1, 2, \ldots, n-k);$$

then, by Theorem 1 of Remark [3], for the roots of the polynomials $\{F_{\cos}[\Phi^{(k)}_j(x)]\}^n_{j=0}$ the inequalities

$$\gamma^{(m)}_{0,k} < \gamma^{(m)}_{1,k} < \cdots < \gamma^{(m)}_{n,k} \quad (m = 1, 2, \ldots, n-k)$$

hold. In the notation adopted,

$$\gamma^{(i)}_{m,k} = \alpha^{(k)}_{i+1}, \qquad \gamma^{(i)}_{0,k} = \beta_i \quad (i = 1, 2, \ldots, n-k).$$

In the paper [2] by V. A. Gusev it was shown that all the numbers

$$\Phi^{(k)}_0(\theta^{(k)}_i),\ \Phi^{(k)}_1(\theta^{(k)}_i), \ldots, \Phi^{(k)}_n(\theta^{(k)}_i)$$

have the same sign (plus for $i = n-k+1, n-k-1, n-k-3, \ldots$, and minus for $i = n-k, n-k-2, n-k-4, \ldots$). From this result, with the application of Theorem 1 [3], we find that all the numbers

$$\{F_{\cos}[\Phi^{(k)}_j(x)]\}_{\varphi=\xi^{(k)}_m} \quad (j = 0, 1, \ldots, n;\ m = 1, 2, \ldots, n-k+1)$$

have the same sign (plus for odd m and minus for even m). Consequently, if

$$\varphi_0 \in [\alpha^{(k)}_i, \beta^{(k)}_i],$$

then all the numbers

$$\{F_{\cos}[\Phi^{(k)}_j(x)]\}_{\varphi=\varphi_0} \quad (j = 0, 1, \ldots, n)$$

have the same sign, and for

$$\varphi_0 = \alpha^{(k)}_m \quad (m = 2, 3, \ldots, n-k+1)$$

only

$$F_{\cos}[\Phi_n^{(k)}(x)] = 0,$$

whereas for

$$\varphi_0 = \beta_m^{(k)} \quad (m = 1, 2, \dots, n-k)$$

only

$$F_{\cos}[\Phi_0^{(k)}(x)] = 0.$$

Thus, for

$$\varphi \in [\alpha_i^{(k)}(\rho); \beta_i^{(k)}(\rho)]$$

the loads $\delta_0, \delta_1, \dots, \delta_n$ have alternating signs, and condition 2) of the criterion is fulfilled. At the right endpoints

$$\{\alpha_m^{(k)}\}_{m=2}^{n-k+1}$$

the load δ_n of the node $\tau_n = 1$ drops out, i.e. $\delta_n = 0$, and at the left endpoints

$$\{\beta_m^{(k)}\}_{m=1}^{n-k}$$

the load δ_0 of the node τ_0, i.e. $\delta_0 = 0$, drops out. Consequently, at these points the servicing of the functional $F_{z,\cos}^{(k)}$ by the polynomials $\pm T_n(x)$ begins or ends. The polynomial $+T_n(x)$ services the functional $F_{z,\cos}^{(k)}$ in the segments

$$[\alpha_1^{(k)}, \beta_1^{(k)}],\ [\alpha_3^{(k)}, \beta_3^{(k)}], \dots,$$

and the polynomial $-T_n(x)$ in the segments

$$[\alpha_2^{(k)}, \beta_2^{(k)}],\ [\alpha_4^{(k)}, \beta_4^{(k)}], \dots .$$

In the case $k = n$, the formula for the loads has the form ([2])

$$\delta_j = (-1)^{n-j} \times n! \Big/ \prod_{i \neq j} |\tau_j - \tau_i| \quad (j = 0, 1, \dots, n),$$

and for $\varphi \in [0, \pi]$ the functional $F_{z,\cos}^{(n)}$ is serviced by the polynomial $+T_n(x)$.

Theorem 2. *In the half-circle $\rho \geq \rho_{1,k}$, for the functional (3) there are $n+1$ Chebyshev arcs*

$$[\chi_m^{(k)}(\rho); \omega_m^{(k)}(\rho)] \quad (m = 1, 2, \dots, n+1);$$

the endpoints $\chi_2^{(k)}, \chi_3^{(k)}, \dots, \chi_{n+1}^{(k)}$ are the roots of the equation

$$F_{\varphi,\cos}^{(k)}\left[\frac{R_{n+1}(x)}{x-1}\right] = 0$$

and $\chi_1^{(k)} = 0$; the endpoints $\omega_1^{(k)}, \omega_2^{(k)}, \dots, \omega_n^{(k)}$ are the roots of the equation

$$F_{\varphi,\cos}^{(k)}[R_{n+1}(x)/x] = 0$$

and $\omega_{n+1} = \pi$.

The course of the proof of this theorem is the same as that of Theorem 1. Let us note only that the proof is based on the application of Lemma 2.

Theorem 3. *In the half-circle* $\rho \geq \rho_{0,k}$, *between the Chebyshev arcs there are open arcs*

$$(\beta_m^{(k)}(\rho); \alpha_{m+1}^{(k)}(\rho)) \quad (k = 1, 2, \dots, n-1;\ m = 1, 2, \dots, n-k),$$

at whose points all the polynomials of passport $[n, n, 0]$ *(Zolotarev polynomials, normalized in the corresponding way* ([1],[2])*—denote them by*

$\{Q_n(x, \vartheta)\}$; ϑ is a variable coefficient of the leading coefficient of the polynomials $\{Q_n(x, \vartheta)\}$, and only they, moreover each of them at that point of every interval in which $F_{z,\cos}^{(k)}[R_n(x, \vartheta)] = 0$, where

$$R_n(x, \vartheta) = \prod_{m=1}^{n} (x - \sigma_m) = \sum_{l=0}^{n} r_l x^l;$$

$(\overset{+}{\sigma}_i)$ are the nodes of the polynomial $Q_n(x, \vartheta)$. We shall call these arcs Zolotarev arcs.

Theorem 4. On the semicircle $\rho \geq \rho_{1,k}$, between the Chebyshev arcs there are open Zolotarev arcs $(\omega_m^{(k)}, \chi_{m+1}^{(k)})$ $(m = 1, 2, \dots, n)$, at whose points for the functional (3) all polynomials of passport $[n, n, 0]$, and only they, are extremal; moreover, each of them is at that point of every interval in which

$$F_{\varphi,\cos}^{(k)}[R_n(x, \vartheta)] = 0.$$

Lemma 2 also lies at the basis of the proof of this theorem.

Theorem 5. On the semicircle of radius ρ, as $\rho \to \infty$, both boundaries of the m-th Zolotarev interval $(\beta_m^{(k)}(\rho); \alpha_{m+1}^{(k)}(\rho))$ of the functional (1) tend to the argument $(2m-1)\pi/2(n-k)$ $(m = 1, 2, \dots, n-k)$.

Theorem 6. In the half-plane $\operatorname{Im} z > 0$, the branches—the boundaries of the Chebyshev arcs $\alpha_m^{(k)}(\rho)$ $(m = 2, 3, \dots, n-k+1)$, as ρ varies from $\rho_{0,k}$ to $+\infty$, asymptotically approach the rays issuing from the point $(n-1)/2n$ on the axis Ox, and the branches—the boundaries $\beta_m^{(k)}(\rho)$ $(m = 1, 2, \dots, n-k)$, as ρ varies from $\rho_{0,k}$ to $+\infty$, approach the rays issuing from the point $(n+1)/2n$ on the axis Ox. The angles of inclination of the asymptotes are determined in Theorem 5.

Theorem 7. On the semicircle of radius ρ, as $\rho \to \infty$, both boundaries of the m-th Zolotarev interval $(\omega_m^{(k)}(\rho); \chi_{m+1}^{(k)}(\rho))$ of the functional (3) tend to the argument $(2m-1)\pi/2n$ $(m = 1, 2, \dots, n)$.

Theorem 8. In the half-plane $\operatorname{Im} z > 0$, the branches–the boundaries of the Chebyshev arcs of the functional (3) $\chi_m^{(k)}(\rho)$ $(m = 2, 3, \dots, n+1)$, as ρ varies from $\rho_{1,k}$ to $+\infty$, asymptotically approach the rays issuing from the point

$$\left(\frac{n-1}{n}\right)^k \frac{n+1}{2n}$$

on the axis Ox, and the branches–the boundaries $\omega_m^{(k)}(\rho)$ $(m = 1, 2, \dots, n)$, as ρ varies from $\rho_{1,k}$ to $+\infty$, approach the rays issuing from the point

$$\left(\frac{n-1}{n}\right)^k \frac{n+1}{2n}$$

on the axis Ox. The angles of inclination of the asymptotes are determined in Theorem 7.

Corollary 1. The sum of the Zolotarev arcs of the functional (1) on the semicircle of radius ρ tends to $1/n \sin(\pi/2(n-k))$ as $\rho \to \infty$.

Corollary 2. The sum of the Zolotarev arcs of the functional (3) on the semicircle of radius ρ tends to $\left((n-1)/n\right)^k/n \sin(\pi/2n)$ as $\rho \to \infty$.

For lack of space we do not present the corresponding results for the functionals $F_{z,\sin}^{(k)}$ and $F_{\varphi,\sin}^{(k)}$. We note only that for $\rho \geq \rho_{0,k}$, for $F_{z,\cos}^{(k)}$ and $F_{z,\sin}^{(k)}$, the Zolotarev arcs do not overlap. Then, if by $E_1(\rho)$ and $E_2(\rho)$ we denote the set of points of the Zolotarev arcs of the functionals $F_{z,\cos}^{(k)}$ and $F_{z,\sin}^{(k)}$, respectively, then at the points of the set $[0,\pi] \setminus E_1(\rho) \cup E_2(\rho)$ the inequality

$$|\mathcal{L}_n^{(k)}(z_0)| \leq M_{\mathcal{L}} \cdot |T_n^{(k)}(z_0)|$$

holds.

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