ON STRONG ANALYTICITY AND STRONG QUASIANALYTICITY OF FUNCTIONS OF SEVERAL VARIABLES

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ON STRONG ANALYTICITY AND STRONG QUASIANALYTICITY OF FUNCTIONS OF SEVERAL VARIABLES

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The idea of classifying functions by means of best approximations was first put forward by S. N. Bernstein (^{1,2}). He revealed the connection between the uniform best approximation $E_n(f)$ of a function $f(x)$ and its intrinsic properties, making it possible to define a given class of continuous functions by prescribing the corresponding order of the rate of decrease of $E_n(f)$. Only for analytic functions does their uniform approximation by polynomials of degree n decrease, as n increases, in a geometric progression, and the analytic continuation of the function $f(x)$ is determined as the unique continuation for which this property is preserved. This same property of uniqueness of continuation was extended to the functions called by S. N. Bernstein quasianalytic of class (P), for which approximation of the same order as for analytic functions is possible for a certain subsequence of the natural series of numbers. Subsequently, the properties of quasianalytic functions of one variable were studied by A. F. Timan (^{3}) and others.

A classification of the same kind for continuous functions of a complex variable was given by S. N. Mergelyan (^{4}) for functions analytic in some domain D and continuous in $\overline{D}$. By means of the constructive theory of functions he obtained all the basic properties of continuous, in particular analytic, functions, proceeding directly from the definition of a function through best approximation. Having established the connection between two well-known classes—the quasianalytic classes (P) of S. N. Bernstein and the quasianalytic classes (D) of Denjoy-Carleman—Mergelyan introduces new quasianalytic classes of functions of a complex variable depending on the rate of approximation to them by polynomials of best approximation.

In the present paper the connection is studied between the best polynomial approximation of functions of one and several variables, the p-th power of whose modulus is Lebesgue integrable with a nonnegative weight in a cube D, and the properties of these functions. New classes of strongly analytic and strongly quasianalytic functions are established, possessing the property of uniqueness of their determination when values are prescribed almost everywhere on any cube contained in D, and necessary and sufficient conditions are given for continuation of functions of these classes beyond the cube D. The apparatus for this continuation consists of partial sums of the expansion in a series of Chebyshev polynomials of the function coinciding almost everywhere in D with the given function, which gives, in a certain sense, a stable extrapolation of the function beyond the cube D.

Denote by $L_p(D, \mu)$ the set of all measurable functions $f(x_1, \dots, x_k)$, defined in the cube $D = D(|x_j| \leq 1,\ j = 1, \dots, k)$, for which the p-th power of the modulus is Lebesgue integrable with weight $\mu > 0$.

Let

$$\|f\|_{p,n}^{(D,\mu)} = \inf_{(F_n)} \|f - F_n(x, \alpha)\|_{L_p(D,\mu)}, \quad 1 \leq p \leq \infty, \tag{1}$$

means the best power approximation of a function $f \in L_p(D, \mu)$ by means of algebraic polynomials

$$F_n(x, \alpha) = \sum_{|m| \leq n} \alpha_{m_1 \dots m_k} x_1^{m_1} \cdots x_k^{m_k},$$

where $|m| = m_1 + \dots + m_k$, and for $p = \infty$

$$\|f\|_{p,n}^{(D,\mu)} = \inf_{(F_n)} \operatorname{vrai} \sup_{(D)} |f - F_n(x, \alpha)| \cdot \mu.$$

Definition 1. The class of functions $f \in L_p(D, \mu)$ satisfying the limiting equality

$$\overline{\lim_{n\to\infty}} \left(\|f\|_{p,n}^{(D,\mu)}\right)^{1/n} = s \quad (s < 1), \tag{2}$$

will be called the **class of strongly analytic functions**, and we shall denote it by $\mathfrak{M}^k(D, \mu, p)$. In the case $s = 0$, this class will be called the **class of strongly entire functions** and denoted by $\mathfrak{M}_0^k(D, \mu, p)$.

For $p = \infty$, the class $\mathfrak{M}^k(D, \mu, p)$ contains the class of analytic functions, which we shall denote by $A^k(D, \mu)$.

Since the condition

$$\lim_{n\to\infty} \|f\|_{p,n}^{(D,\mu)} = 0 \tag{3}$$

is necessary and sufficient for $f \in L_p(D, \mu)$, it seems natural to introduce a new notion of strong quasianalyticity of a function.

Definition 2. The class of functions $f \in L_p(D, \mu)$ determined by the condition

$$\varliminf_{n\to\infty} \left(\|f\|_{p,n}^{(D,\mu)}\right)^{1/n} = s \quad (s < 1), \tag{4}$$

will be called the **class of strongly quasianalytic functions**, and we shall denote it by $\mathfrak{N}^k(D, \mu, p)$. In the case $s = 0$, this class will be called the **class of strongly quasientire functions** and denoted by $\mathfrak{N}_0^k(D, \mu, p)$.

For $p = \infty$, the class $\mathfrak{N}^k(D, \mu, p)$ contains the class of quasianalytic functions, which for $k = 1$ was first introduced by S. N. Bernstein.

From (2) and (4) it follows that for any function f from the class $\mathfrak{N}^k(D, \mu, p)$, or else from the class $\mathfrak{M}^k(D, \mu, p)$, there exists an increasing sequence of natural numbers $\{n_i\}_1^\infty$ such that the best power approximation of the function on the cube D by polynomials $F_n(x, \alpha)$ in the variables $x_1, \dots, x_k$ of degree n_i, for all values of i, satisfies the inequality

$$\|f\|_{p,n_i}^{(D,\mu)} < M s_1^{n_i}, \tag{5}$$

where M and s_1 do not depend on n_i, $s < s_1 < 1$. Each such sequence $\{n_i\}_1^\infty$ generates its own class of strongly quasianalytic functions $\mathfrak{N}^k(D, \mu, p)_{(n_i)}$, and correspondingly a class of strongly analytic functions $\mathfrak{M}^k(D, \mu, p)_{(n_i)}$. Obviously, the class $\mathfrak{N}^k(D, \mu, p)$ is the class of all possible classes $\mathfrak{N}^k(D, \mu, p)_{(n_i)}$, and the class $\mathfrak{M}^k(D, \mu, p)$ is the class of classes $\mathfrak{M}^k(D, \mu, p)_{(n_i)}$.

On the basis of (5), to the function $f(x) \in \mathfrak{N}^k(D, \mu, p)$ there corresponds on the cube D a unique function

$$\varphi(x) = \sum_{i=1}^{\infty} A_{n_i} T_{n_i}(x),$$

which is the limit of the sequence $\{p_{n_i}(x)\}_1^\infty$ of polynomials of best power approximation of the function $f(x)$ on the cube D. The function $\varphi(x)$ is continuous and almost everywhere coincides with $f(x)$ on the cube D.

Definition 3. Let $f(x) \in \mathfrak{N}^k(D, \mu, p)$, and let $\varphi(x)$ be a function uniquely determined by the sequence $\{n_i\}_1^\infty$ from (5). An elliptic polycylinder T_R^k $(R > 1)$ of the space C_z^k, at all interior points of which the series

$$\varphi(z) = \sum_{i=1}^{\infty} A_{n_i} T_{n_i}(z) \tag{*}$$

converges absolutely and uniformly, and on whose boundary there is at least one singular point of the expansion $(*)$, will be called an **elliptic polycylinder of strong quasi-analyticity** of the function $f(x) \in \mathfrak{N}^k(D, \mu, p)$.

Theorem 1. *In order that a function $f(x)$ from the space $L_p(D, \mu)$ belong to the strongly quasi-analytic class $\mathfrak{N}^k(D, \mu, p)$, it is necessary and sufficient that there exist a function $\varphi(z)$, coinciding almost everywhere in the cube D with the function $f(x)$ and having, with respect to some monotonically increasing sequence of natural numbers $\{n_i\}_1^\infty$, an expansion of the form $(*)$, converging absolutely and uniformly in the cube D with the rate of the common term of some decreasing geometric progression.*

Functions of the strongly quasi-analytic class are characterized by a property analogous to the uniqueness property of analytic functions; namely, the following holds:

Theorem 2. *A function $f(x) \in \mathfrak{N}^k(D, \mu, p)$ that is equivalent to zero on any interior cube $D_\varepsilon = D_\varepsilon(|x_j| \leq 1 - \varepsilon;\ j = 1, \dots, k)$ $(0 < \varepsilon < 1)$ is equivalent to zero on the whole cube D.*

Let us identify some properties of functions of the strongly quasi-analytic class.

Theorem 3. *Two sequences of increasing natural numbers $\{n_i\}_1^\infty$ and $\{n_i'\}_1^\infty$, connected by the relation*

$$l^{-1} < n_i' n_i^{-1} < l \quad (i = 1, 2, \dots), \tag{6}$$

where $l > 0$ is some constant, cannot generate different classes of strongly quasi-analytic functions on the cube D.

Theorem 4. *Let $\{n_i\}_1^\infty$ be a sequence generating the class $\mathfrak{N}^k(D, \mu, p)_{(n_i)}$. If the ratio n_{i+1}/n_i is unbounded, then the class $\mathfrak{N}^k(D, \mu, p)_{(n_i)}$ necessarily contains functions not belonging to the class $\mathfrak{N}^k(D, \mu, p)$. If, however, this ratio is bounded, then the class $\mathfrak{N}^k(D, \mu, p)_{(n_i)}$ consists solely of functions of the class $\mathfrak{N}^k(D, \mu, p)$.*

Theorem 5. *Every function $f \in L_p(D, \mu)$ is equivalent on the cube D to the sum of two strongly quasi-analytic functions generated by certain increasing sequences of natural numbers $\{n_i\}_1^\infty$ and $\{n_i'\}_1^\infty$.*

For $k = 1,\ p = \infty$, in the space of continuous functions, this theorem was proved by A. I. Markushevich (^{5}).

Theorem 6. *Let $f \in \mathfrak{N}^k(D, \mu, p)_{(n_i)}$. In order that, in the cube $D_1 \supset D$, the function $f \in \mathfrak{N}^k(D_1, \mu, p)_{(n_i)}$, i.e., have a unique strongly quasi-analytic continuation to the cube D_1, it is necessary and sufficient that the polynomials $F_{n_i}(x, \alpha)$ of best degree approximation to the function f in the cube D be polynomials of best degree approximation to the function f also in the cube D_1, and that in the cube $D_\varepsilon = D_\varepsilon(|x_j| \leq 1 + \varepsilon,\ j = 1, \dots, k)$ $(\varepsilon > 0)$, $D \subset D_\varepsilon \subset D_1$, the function $f \in \mathfrak{N}^k(D_\varepsilon, \mu, p)_{(n_i)}$.*

Let us put in correspondence with each function $f \in L_p(D, \mu)$ a function φ, coinciding almost everywhere in D with the function f and having everywhere in

D an expansion in a series in Chebyshev polynomials, converging to φ absolutely and uniformly. When the conditions of Theorem 6 are fulfilled, the partial sums S_{n_i} of the expansion of the function φ determine a unique strongly quasi-analytic continuation of the function $f \in \mathfrak{N}^k(D, \mu, p)_{(n_i)}$ to the cube $D_1 \supset D$. It may happen, however, that the sequence of partial sums $\{S_n\}_1^\infty$ splits into sequences $\{S_{n_{ji}}\}_1^\infty$ $(j = 1, \ldots, N;\ N \geq 1)$, generating-

⋯in the cube $D_1 \supset D$, different classes of strongly quasianalytic functions. In this case we shall speak of a special strongly quasianalytic continuation of a function $f \in \mathfrak{N}^k(D, \mu, p)_{(n_i)}$ beyond the cube D to the cube D_1. Repeating the process of strongly quasianalytic continuation, we shall reach the boundary of the domain of strong quasianalyticity of the function, if the function f is not a strong quasicell. In this case the function $f \in \mathfrak{N}^k(D, \mu, p)_{(n_i)}$ will have as many domains of strong quasianalyticity as there are distinct strongly quasianalytic continuations of it beyond the cube D. If, further, outside the domain of strong quasianalyticity of the function f, containing the cube $D^* \supset D$, the partial sums S_n of the expansion in Chebyshev polynomials of the function φ, which coincides with f almost everywhere in D^*, converge strongly, with respect to some sequence $\{n_i'\}_1^\infty$ of increasing natural numbers, to a certain function $f_1 \in L_p(D_1^*, \mu)$, $D_1^* \supset D^*$, we shall speak of a pseudo-strongly quasianalytic continuation of the function, which also may be single-valued or multivalued. Carrying out successively the process of such continuation, we shall encounter the boundary of pseudo-strongly quasianalytic continuation, beyond which the sums S_n have no limit, for any subsequence of the natural number series, in the sense of strong convergence.

In the presence of one or another of the continuations indicated above of a function $f \in \mathfrak{N}^k(D, \mu, p)_{(n_i)}$, the strong extrapolation of the function f beyond the cube D is defined. The extrapolation so defined is stable in the sense that any change in the cube D of the values of the function f on a set of measure zero does not affect the convergence of the extrapolation.

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Note: Figure translations are in progress. See original paper for figures.