THEORY OF A NEUTRAL MASSIVE TENSOR FIELD WITH SPIN 2

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V. I. OGIEVETSKII, I. V. POLUBARINOV

THEORY OF A NEUTRAL MASSIVE TENSOR FIELD WITH SPIN 2

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1. An interacting symmetric tensor field $h^{\mu\nu}$ ($h^{\nu\mu} = h^{\mu\nu}$) is capable of describing spins 2, 1, and two spins 0 ([1]). In the present note we shall construct such a theory of an interacting massive tensor field in which from the equations of motion there follows one most general linear supplementary condition—the Hilbert-Lorentz type condition

$$m^2(\partial_\mu h^{\mu\nu} + q\partial_\nu h^{\mu\mu}) = 0, \tag{1}$$

where m is the mass of the field $h^{\mu\nu}$, and q is a certain arbitrary number. In such a theory the field $h^{\mu\nu}$ will describe only spins 2 and 0, while the other spin 0 and spin 1 will be excluded. The exclusion of the latter is necessary, since the sign of the energy for spin 1 is opposite to the sign of the energy for spins 2 and 0.

2. We shall consider the interaction of the tensor field with itself and with a scalar field φ. Interactions with other fields are investigated in a completely analogous way ([1]), and we omit them here for brevity.

In ([1]) it was proved that from the equations of motion there will follow the supplementary condition (1), if for the Lagrangian density the identity holds

$$\int d^4x\, \delta^*\mathcal{L} \equiv -2m^2 \int d^4x\, \partial_\mu \lambda^\nu \,(h_p^{\mu\nu} + q\delta^{\mu\nu} h_p^{\rho\rho})\,, \tag{2}$$

where $\delta^*\mathcal{L}$ is the result of varying $\mathcal{L}$ on the following class of variations of the fields $h_p^{\mu\nu}$ and φ:

$$\delta^* h_p^{\mu\nu} = \partial_\mu\lambda^\nu + \partial_\nu\lambda^\mu + p\delta^{\mu\nu}\partial_\sigma\lambda^\sigma +$$

$$+a(\partial_\sigma\lambda^\nu h_p^{\mu\sigma} + \partial_\sigma\lambda^\mu h_p^{\sigma\nu} + p\partial_\sigma\lambda^\sigma h_p^{\mu\nu} - \lambda^\sigma\partial_\sigma h_p^{\mu\nu}), \tag{3}$$

$$\delta^* \varphi = -a\lambda^\sigma \partial_\sigma \varphi. \tag{4}$$

In (3), (4), $\lambda^\mu(x)$ is a completely arbitrary infinitesimal dimensionless 4-vector function, a and s are constants of dimension cm, p is a dimensionless constant characterizing the "weight." As in the massless case, it is convenient to pass to the quantity $g_p^{\mu\nu} = \delta^{\mu\nu} + a h_p^{\mu\nu}$, whose variation is written in the form

$$\delta^* g_p^{\mu\nu} = a(\partial_\sigma \lambda^\nu g_p^{\mu\sigma} + \partial_\sigma \lambda^\mu g_p^{\sigma\nu} + p\partial_\sigma \lambda^\sigma g_p^{\mu\nu} - \lambda^\sigma \partial_\sigma g_p^{\mu\nu}). \tag{5}$$

3. It is possible to express the spin constraint in terms of a more general identity than (2). Namely, as the initial one one may take the identity

$$\int d^4x\, \delta^* \mathcal{L} \equiv -2m^2 \int d^4x\, \partial_\mu \lambda^\nu \left[f^{\mu\nu}\left(h_p^{\alpha\beta}\right) + q\delta^{\mu\nu} f^{\rho\rho}\left(h_p^{\alpha\beta}\right)\right], \tag{6}$$

where $f^{\mu\nu}$ is any function of $h_p^{\alpha\beta}(x)$ admitting inversion. In fact, one may take $f^{\mu\nu}$ as a new field variable, for which from the identity (6) there follows the linear supplementary condition (3) $\partial_\mu f^{\mu\nu} + q\partial_\nu f^{\mu\mu} = 0$. In terms of $h_p^{\mu\nu}$ it would have a highly nonlinear structure. By $h_p^{\mu\nu}$ is meant the quantity varied according to the linear law (3). Obviously, the field $f^{\mu\nu}$ will vary, generally speaking–

according to a nonlinear law. Thus, passing to identity (6), we cover a broad class of nonlinear variations.

4. Remove the integration in identity (6). Then it takes the form:

$$\delta^* \mathcal{L} \equiv -2m^2 \partial_\mu \lambda^\nu \left(f^{\mu\nu} + q\delta^{\mu\nu} f^{\rho\rho}\right) + \partial_\mu X^\mu, \tag{7}$$

where X^μ is some 4-vector, also subject to determination. From the condition of compatibility with Lorentz invariance (for example, for $\lambda^\mu = \text{const}$ the transformations (3), (4) formally coincide with translations), one may conclude that $X^\mu = -a\lambda^\mu \mathcal{L}$. Let us decompose the Lagrangian into two parts

$$\mathcal{L} = \mathcal{L}'' + m^2 \mathcal{L}_m, \tag{8}$$

where $\mathcal{L}'' = \mathcal{L}\big|_{m=0}$ is the massless part, satisfying the identity

$$\delta^* \mathcal{L}'' \equiv -a\partial_\mu(\lambda^\mu \mathcal{L}) \tag{9}$$

and, as shown in (1), coinciding with the Lagrangian density of Einstein's theory

$$\mathcal{L}'' = \left(-\frac{2}{a^2} R - \frac{1}{2} g^{\lambda\rho} \partial_\lambda \varphi\, \partial_\rho \varphi - \frac{\mu^2}{2} \varphi\varphi\right) g_0^{-1/2}. \tag{10}$$

From (7)–(9) it follows that

$$\delta^* \mathcal{L}_m \equiv -2\partial_\mu \lambda^\nu \left(f^{\mu\nu} + q\delta^{\mu\nu} f^{\rho\rho}\right) - a\partial_\mu (\lambda^\mu \mathcal{L}_m). \tag{11}$$

5. We shall assume that $\mathcal{L}_m$ depends only on $g_p^{\mu\nu}$ (or, what is the same, on $h_p^{\mu\nu}$ or $f^{\mu\nu}$), but does not depend on other fields or on derivatives of the tensor field: $\mathcal{L}_m = \mathcal{L}_m(g_p^{\mu\nu})$. Then identity (11) will be equivalent to two identities. The first of them (corresponding to $\lambda^\alpha = \text{const}$) simply expresses translational invariance. The second, obtained for $\lambda^\alpha = \delta^{\alpha\rho} x^\mu$, has the form

$$2a \frac{\delta \mathcal{L}_m}{\delta g_p^{\rho\alpha}} g_p^{\mu\alpha} + ap\, \delta^{\mu\rho} \frac{\delta \mathcal{L}_m}{\delta g_p^{\alpha\beta}} g_p^{\alpha\beta} \equiv -2\left(f^{\mu\rho} + q\delta^{\mu\rho} f^{\nu\nu}\right) - a\delta^{\mu\rho} \mathcal{L}_m. \tag{12}$$

Contracting in (12) the indices μ and ρ and using the result obtained, we eliminate from (12)

$$\frac{\delta \mathcal{L}_m}{\delta g_p^{\alpha\beta}} g_p^{\alpha\beta}.$$

After this, identity (12) can be brought to the form:

$$\delta\Big(\mathcal{L}_m g_p^{1/2(1+2p)}\Big) \equiv -\frac{1}{a} g_p^{1/2(1+2p)}\, \delta g_p^{\mu\nu} g_{p\nu\lambda} \left[f^{\lambda\mu} + \frac{2q-p}{2(1+2p)} \delta^{\lambda\mu} f^{\rho\rho}\right]. \tag{12'}$$

6. This identity determines $\mathcal{L}_m$ in any theory of a tensor field in which the supplementary condition (1) is satisfied for $f^{\mu\nu}$. The function $f^{\mu\nu}(g_p^{\alpha\beta})$ may be chosen in many ways. We shall dwell on the special form

$$f_p^{\mu\nu} = \frac{1}{na}\left(g_p^{n\mu\nu} - \delta^{\mu\nu}\right), \tag{13}$$

where $g_p^{n\mu\nu}$ are the elements of the matrix

$$\|g_p^{\mu\nu}\|^n = \|\delta^{\mu\nu} + ah_p^{\mu\nu}\|^n = \delta^{\mu\nu} + nah_p^{\mu\nu} + \frac{n(n-1)}{2} a^2 h_p^{\mu\lambda} h_p^{\lambda\nu} + \cdots,$$

with $g_p^{0\mu\nu} = \delta^{\mu\nu}$. This form includes the field variables usually used: for $n = 1,\ p = 0$,

$$f^{\mu\nu} = h_0^{\mu\nu} = \frac{1}{a}(g^{\mu\nu} - \delta^{\mu\nu}),$$

where $g^{\mu\nu}$ is the ordinary contravariant "metric" tensor; for $n = -1,\ p = 0$, $f^{\mu\nu}$ is related to the covariant "metric" tensor; for $n = \frac{1}{2}$,

$$f_p^{\mu\nu} = \frac{2}{a}\left(r_p^{\mu\nu} - \delta^{\mu\nu}\right),$$

where $r_p^{\mu\nu}$ are the elements of the matrix $\|g_p^{\mu\nu}\|^{1/2}$. The "root" of $g^{\mu\nu}$ plays an important role in the study of spinors ([2]).

Substituting (13) into (12′), we obtain

$$\delta(\mathcal{L}_m g_p^{1/2(1+2p)}) = -\frac{1}{na^2} g_p^{1/2(1+2p)} \left\{ \delta g_p^{\mu\nu} g_p^{(n-1)\mu\nu} + \frac{2q-p}{2(1+2p)} \delta g_p^{\mu\nu} g_{p\mu\nu} g_p^{n\lambda\lambda} - \frac{1+4q}{1+2p} \delta g_p^{\mu\nu} g_{p\mu\nu} \right\}. \tag{14}$$

This equation is easily integrated when

$$q = (np+1)/2n. \tag{15}$$

In this case we find

$$\mathcal{L}_m = C g_p^{-1/2(1+2p)} - \frac{1}{n^2a^2}\{g_p^{n\mu\mu} - 2[(1+2p)n+2]\}.$$

The constant C can be determined from the requirement that, for $a=0$, the mass term be quadratic in $f^{\mu\nu}$. This gives $C = -2(1+2p)/na^2$. Thus, we obtain the family of mass terms

$$m^2\mathcal{L}_m = \frac{m^2}{na^2}\left\{-2(1+2p)g_p^{-1/2(1+2p)} - \frac{1}{n}g_p^{n\mu\mu} + \frac{2}{n}[(1+2p)n+2]\right\} =$$

$$= -\frac{m^2}{4n(1+2p)}[f_1^2 + 2n(1+2p)f_2] +$$

$$+\frac{am^2}{24n(1+2p)^2} \times [f_1^3 + 6n(1+2p)f_1f_2 + 8n^2(1+2p)^2 f_3] + O(a^2),$$

where $f_1 = f_p^{\alpha\alpha}$, $f_2 = f_p^{\alpha\beta} f_p^{\beta\alpha}$, $f_3 = f_p^{\alpha\beta} f_p^{\beta\gamma} f_p^{\gamma\alpha}$. The equation of motion for the massive tensor field is written in the form

$$g_p^{-(p+1)/2(1+2p)}\left\{-\frac{2}{a^2}\left(R_{\mu\nu} - \frac{p+1}{2(1+2p)} g_{p\mu\nu} g_p^{\alpha\beta} R_{\alpha\beta}\right) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \right.$$

$$\left. + \frac{g_{p\mu\nu}}{4(1+2p)}\left[(p+1)g_p^{\lambda\rho}\partial_\lambda\varphi\partial_\rho\varphi + \mu^2 g_p^{1/2(1+2p)} g_{p\mu\nu}\varphi\varphi\right]\right\} +$$

$$+\frac{m^2}{na^2}\left[g_p^{-1/2(1+2p)} g_{p\mu\nu} - g_p^{(n-1)\mu\nu}\right] = 0. \tag{17}$$

The Hilbert-Lorentz condition following from equation (17), with allowance for (15), takes the form

$$m^2\left(\partial_\mu f_p^{\mu\nu} + \frac{np+1}{2n}\partial_\nu f_p^{\mu\mu}\right) = 0. \tag{18}$$

For $p \neq -(n+2)/2n$ ($q \neq -1/4$), condition (18) can be reduced to the Lorentz condition

$$m^2\partial_\mu f'^{\mu\nu}_{p'} = 0. \tag{18'}$$

For this it is necessary to make a change of the field variables

$$f'^{\mu\nu}_{p'} = f_p^{\mu\nu} + \frac{np+1}{2n}\delta^{\mu\nu} f_p^{\rho\rho}. \tag{19}$$

The parameters p' and p of the additive parts of the variations $f'^{\mu\nu}_{p'}$ and $f_p^{\mu\nu}$, according to (13) and (19), are related by

$$p' = 2\left(p + \frac{n+1}{2n}\right)^2 - \frac{n^2+1}{2n^2}. \tag{20}$$

In these new variables the Lagrangian density and the equations of motion can be left in the form (10), (16), and (17), putting there

$$g_p^{n\mu\nu} = \delta^{\mu\nu} + naf_p^{\mu\nu} = \delta^{\mu\nu} + na\left\{f'^{\mu\nu}_{p'} - \frac{np+1}{2[(1+2p)n+2]}\delta^{\mu\nu} f'^{\rho\rho}_{p'}\right\}, \tag{21}$$

$$g_p^{\mu\nu} = \delta^{\mu\nu} + af_p^{\mu\nu} + \frac{1-n}{2}a^2 f_p^{\mu\lambda} f_p^{\lambda\nu} + \frac{(1-n)(1-2n)}{6}a^3 f_p^{\mu\lambda} f_p^{\lambda\rho} f_p^{\rho\nu} + \cdots.$$

Thus, for the massive tensor field we have found a two-parameter family of inequivalent theories, characterized by the values-

⋯by the values of the parameters p' and n. In each such theory the field $j^{\mu\nu}_{p'}$ is a superposition of spin 2 with mass m and spin 0 with mass $m_0 = m\sqrt{-\frac{1+2p'}{2+p'}}$. This is seen from the free equation for the component of the field with spin 0, which is in fact represented by $f^{\mu\mu}_{p'}$,

$$\left(\frac{2+p'}{1+2p'}\Box + m^2\right) f^{\mu\mu}_{p'} = 0. \tag{22}$$

It follows from this that the values of p' are confined to the range $-2 \leq p' < -1/2$ (otherwise the mass of the scalar component of the field $f^{\mu\nu}$ would be imaginary). In the case of interaction, for $n^2 > 1/3$ the bounds become narrower, since from (20) it follows that $p' \geq -(n^2+1)/2n^2$. For $p' = -1$ the masses of the components with spins 0 and 2 coincide, while as p' is increased to $-1/2$ the mass of the scalar component decreases monotonically to zero, without reaching the lower bound. The most interesting case is $p' = -2$ (possible when $n^2 \leq 1/3$), in which the emitted particles have only spin 2 (just as massless gravitons do).

It remains to discuss the case $q = (np+1)/2n = -1/4$, when the Hilbert-Lorentz condition (18) cannot be reduced to the Lorentz condition. In this case the field $f^{\mu\nu}$ has components with spin 2 and mass m, and with spin 0 and zero mass. It is curious that, despite the presence of mass terms for the tensor and all other fields, for $q = -1/4$ (and only for $q = -1/4$) the theory is invariant with respect to the 15-parameter conformal group C_4. At the same time, for the remaining q there is invariance only with respect to the inhomogeneous Lorentz group. These statements follow from the fact that identity (6) becomes a condition of invariance with respect to the transformations (3), (4), when λ^ν satisfy the equation $\partial_\mu \lambda^\nu + \partial_\nu \lambda^\mu + q\delta^{\mu\nu}\partial_\sigma \lambda^\sigma = 0$. For $q \neq -1/4$ the solution is 10-parameter: $a\lambda^\nu = c^\nu + \omega^{\nu\mu}x^\mu$ ($c^\nu = \text{const}$, $\omega^{\nu\mu} = -\omega^{\mu\nu} = \text{const}$), and the transformations (3), (4) turn out to be transformations of the 10-parameter inhomogeneous Lorentz group. For $q = -1/4$ the solution is broader: $a\lambda^\nu = c^\nu + \omega^{\nu\mu}x^\mu + bx^\nu - \frac{1}{2}d^\nu x^\mu x^\mu + d^\mu x^\mu x^\nu$ ($b = \text{const}$, $d^\nu = \text{const}$), so that the transformations (3), (4) become transformations of the 15-parameter conformal group.

Thus, the principle of restriction by spin for a massive tensor field gives a set of theories, whereas for a massless tensor field a unique theory was obtained –the Einstein theory. Equation (17) differs from the Einstein equation with cosmological term (3) by the last term on the left-hand side. This addition violates the general covariance of the theory and the equivalence principle. At the same time, the equality of inertial and "gravitational" masses remains valid.

Naturally, in these theories the static potential is similar to the Yukawa potential and not to the Coulomb potential, while the cosmological term does not give solutions with Yukawa behavior. In principle such theories may be useful if the graviton has a mass (of course, a very small one), and for describing some other heavy particles with spin 2, if they exist.

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Note: Figure translations are in progress. See original paper for figures.