ON AN ASYMPTOTIC EXPANSION IN THE CENTRAL LIMIT THEOREM

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Abstract

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MATHEMATICS

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ON AN ASYMPTOTIC EXPANSION IN THE CENTRAL LIMIT THEOREM

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Let $X_1, X_2, \ldots$ be a sequence of mutually independent identically distributed random variables with mathematical expectation $m = EX_1$ and positive variance $\sigma^2 = E(X_1 - m)^2$. Introduce the notation

$$F_n(x) = P\left\{\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(X_j - m) < x\right\}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Theorem. Suppose that the following conditions are satisfied:

1) $E|X_1|^k < \infty$ for some integer $k \geq 3$;
2)
$$\lim_{|t|\to\infty} \left|Ee^{itX_1}\right| < 1.$$

Then there exists a function $\varepsilon(n)$, independent of x, such that

$$\lim_{n\to\infty} \varepsilon(n) = 0$$

and

$$\left|F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}}\right| \leq \frac{\varepsilon(n)}{(1+|x|^k)n^{(k-2)/2}}$$

for all x $(-\infty < x < \infty)$. The functions $P_\nu(-\Phi)$ are defined in the same way as in ([1,4]).

This theorem is a refinement of a theorem of Esseen ([1,2]), according to which, under the conditions of the theorem formulated above, the relation

$$F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}} = o\left(\frac{1}{n^{(k-2)/2}}\right)$$

holds as $n \to \infty$, uniformly with respect to x $(-\infty < x < \infty)$.

V. V. Petrov ([3]) earlier found conditions necessary and sufficient for the validity of the relation

$$\left|\frac{d}{dx}\left(F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}}\right)\right| \leq \frac{\varepsilon(n)}{(1+|x|^k)n^{(k-2)/2}}$$

for all x $(-\infty < x < \infty)$, where $\varepsilon(n)$ does not depend on x and

$$\lim_{n\to\infty}\varepsilon(n) = 0.$$

From the theorem stated above there follows directly a series of consequences concerning the global form of integral limit theorems.

Under the conditions of the theorem, the following assertions are valid:

1. For any $p > 1/k$, as $n \to \infty$ we have

$$\int_{-\infty}^{\infty}\left|F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}}\right|^p dx = o\left(\frac{1}{n^{(k-2)p/2}}\right).$$

2. For any $p \geqslant 1$, as $n \to \infty$ we have

$$\int_{-\infty}^{\infty}|F_n(x) - \Phi(x)|^p dx = \int_{-\infty}^{\infty}\left|\sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}}\right|^p dx + o\left(\frac{1}{n^{(k+p-3)/2}}\right).$$

3. For any $p \geqslant 1$, as $n \to \infty$ we have

$$\|F_n(x) - \Phi(x)\| = \left\|\sum_{\nu=1}^{k-2}\frac{P_\nu(-\Phi)}{n^{\nu/2}}\right\| + o\left(\frac{1}{n^{(k-2)/2}}\right).$$

Here

$$\|u(x)\| = \left[\int_{-\infty}^{\infty}|u(x)|^p dx\right]^{1/p}$$

for any function $u(x) \in L_p(-\infty, \infty)$.

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4. V. V. Petrov, *Vestnik Leningrad Univ.*, No. 19, 150 (1962).

Note: Figure translations are in progress. See original paper for figures.