SOME THEOREMS ON SOLUTIONS OF A PROBLEM IN METEOROLOGY

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.946

MATHEMATICS

G. V. DEMIDOV

SOME THEOREMS ON SOLUTIONS OF A PROBLEM IN METEOROLOGY

(Presented by Academician S. L. Sobolev on 24 V 1965)

Consider the following problem: the functions u, v, H satisfy the system of equations

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-lv+\frac{\partial H}{\partial x}=0,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+lu+\frac{\partial H}{\partial y}=0,\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-\frac{\partial}{\partial p}\frac{p^2}{\mu}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial H}{\partial p}=0\end{gathered} \tag{1}$$

for

$$t>0, \qquad 0<p_0<p<P;$$

the boundary conditions

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial H}{\partial p}=0 \quad \text{for } p=p_0,$$

$$p\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial H}{\partial p}+\alpha\frac{\partial H}{\partial t}=0 \quad \text{for } p=P; \tag{2}$$

the initial conditions

$$u(x,y,0,p)=u^0(x,y,p), \qquad v(x,y,0,p)=v^0(x,y,p),$$

$$H(x,y,0,p)=H^0(x,y,p). \tag{3}$$

Here l is an analytic function of x, y; α is an analytic function of x, y, t, with $\alpha \geq \alpha_0 > 0$; μ is a continuously differentiable function of p, with $0 < \mu_0 \leq \mu \leq \mu_1$; $p_0, P, \alpha_0, \mu_0, \mu_1$ are constants.

Definition 1. We shall say that a function $f(x_1, \ldots, x_n, p)$, analytic in $x_1, \ldots, x_n$ in a neighborhood of the point $(x_1^0, \ldots, x_n^0)$, admits a majorant uniform in p in the interval $p_0 < p < P$ if it expands in the series

$$f(x_1, \ldots, x_n, p) = \sum_{k_i=0}^{\infty} f_{k_1 \ldots k_n}(p)(x_1 - x_1^0)^{k_1} \cdots (x_n - x_n^0)^{k_n}$$

and there exist constants $a_{k_1 \ldots k_n}$ such that

$$\left| f_{k_1 \ldots k_n}(p) \right| \leq a_{k_1 \ldots k_n} \quad \text{for } p_0 < p < P,$$

and the series

$$\sum_{k_i=0}^{\infty} a_{k_1 \ldots k_n}(x_1 - x_1^0)^{k_1} \cdots (x_n - x_n^0)^{k_n}$$

converges in some neighborhood of the point $(x_1^0, \ldots, x_n^0)$.

Theorem 1. If the initial data u^0, v^0, H^0 are functions analytic in x, y in a neighborhood of the point (x_0, y_0), such that $u^0, v^0, H^0, \partial u^0/\partial p, \partial v^0/\partial p, \partial H^0/\partial p, \partial^2 H^0/\partial p^2$ admit majorants uniform in p in the interval $p_0 < p < P$, then there exists a unique-

···solution of problem (1)–(3), analytic in x, y, t in a neighborhood of the point $(x_0, y_0, 0)$, such that $u, v, H, \partial u/\partial p, \partial v/\partial p, \partial H/\partial p, \partial^2 H/\partial p^2$ admit, in a neighborhood of this point, majorants uniform in p in the interval $p_0 < p < P$.

We shall outline the proof. In addition to the functions u, v, H, introduce the functions $u_p = \partial u/\partial p$, $v_p = \partial v/\partial p$, $H_p = \partial H/\partial p$, and $H_{pp} = \partial^2 H/\partial p^2$. The system (1) can be solved with respect to the time derivatives of the functions u, v, H_{pp}. Differentiating the first two equations with respect to p, we obtain equations solved with respect to $\partial u_p/\partial t$ and $\partial v_p/\partial t$. Using the third equation of system (1) and the boundary conditions (2), we obtain equations for H_p and H.

$$\frac{\partial H_p}{\partial t} = -u\frac{\partial H_p}{\partial x} - v\frac{\partial H_p}{\partial y} + \frac{\mu}{p^2}\int_{p_0}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dp,$$

$$\frac{\partial H}{\partial t} = \int_{P}^{p} \left[-u\frac{\partial H_p}{\partial x} - v\frac{\partial H_p}{\partial y} + \frac{\mu}{p^2}\int_{p_0}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dp\right] dp - \frac{\mu(P)}{\alpha P}\int_{p_0}^{P} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dp.$$

The obtained system of equations (denote it by $(1')$) makes it possible to compute uniquely all derivatives with respect to x, y, t as functions of p at the point $(x_0, y_0, 0)$. The problem is therefore reduced to proving the convergence of formal series. Note that system $(1')$ contains derivatives with respect to x and y only of first order and contains no derivatives with respect to p. The integration with respect to p present in two equations can increase the coefficients of the series in x, y, t by no more than a bounded factor. Using this circumstance, it is not difficult to construct a Cauchy–Kovalevskaya system that will be a majorant with respect to system $(1')$. The solutions of this system will not depend on p, whence the assertion of Theorem 1 follows regarding majorants uniform in p.

In the subsequent considerations we weaken the smoothness requirements on α to the requirement: α and $\partial\alpha/\partial t$ are continuous for $t \geqslant 0$. Let (u_1, v_1, H_1) and (u_2, v_2, H_2) be two different solutions of problem (1)–(3) such that all derivatives entering into the equations of system (1) are continuous. The functions u, v, H, defined by the formulas

$$u = (u_1 - u_2)e^{-\lambda t}, \qquad v = (v_1 - v_2)e^{-\lambda t}, \qquad H = (H_1 - H_2)e^{-\lambda t}, \tag{4}$$

where λ is a positive constant, satisfy the system of equations

$$\frac{\partial u}{\partial t} + u_1\frac{\partial u}{\partial x} + v_1\frac{\partial u}{\partial y} + \lambda u + lv + \frac{\partial u_2}{\partial x}u + \frac{\partial u_2}{\partial y}v + \frac{\partial H}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + u_1\frac{\partial v}{\partial x} + v_1\frac{\partial v}{\partial y} + \lambda v - lu + \frac{\partial v_2}{\partial x}u + \frac{\partial v_2}{\partial y}v + \frac{\partial H}{\partial y} = 0, \tag{5}$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial p}\frac{p^2}{\mu}\left[\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + \lambda\right)\frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x\partial p}u + \frac{\partial^2 H_2}{\partial y\partial p}v\right] = 0,$$

with boundary conditions

$$\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + \lambda\right)\frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x\partial p}u + \frac{\partial^2 H_2}{\partial y\partial p}v = 0 \quad \text{for } p = p_0,$$

$$p\left[\left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + \lambda\right)\frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x\partial p}u + \frac{\partial^2 H_2}{\partial y\partial p}v\right] + \alpha\frac{\partial H}{\partial[[unclear:denominator]]} + \lambda H = 0 \quad \text{for } p = P. \tag{6}$$

and the initial conditions

$$\begin{aligned} u(x,y,0,p) &= u_1^0 - u_2^0, \\ v(x,y,0,p) &= v_1^0 - v_2^0, \\ H(x,y,0,p) &= H_1^0 - H_2^0. \end{aligned} \tag{7}$$

Let Ω be some volume with piecewise smooth boundary S in the space (x, y, t). Multiplying the system of equations (5) scalarly by the vector (u, v, H) and, using the boundary conditions (6), integrating the result over Ω and over p from p_0 to P, we obtain the identity

$$\int_\Omega \left(\frac{\partial Q}{\partial t} + \frac{\partial R}{\partial x} + \frac{\partial L}{\partial y} + K \right) dx\, dy\, dt = 0, \tag{8}$$

where

$$Q = \frac{1}{2} \int_{p_0}^{P} \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] dp + \frac{\alpha P}{2\mu(P)} H^2(P), \tag{9}$$

$$R = \int_{p_0}^{P} \left\{ Hu + \frac{u_1}{2} \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp, \tag{10}$$

$$L = \int_{p_0}^{P} \left\{ Hv + \frac{v_1}{2} \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp, \tag{11}$$

$$\begin{aligned} K = &\left(\lambda\alpha - \frac{1}{2} \frac{\partial \alpha}{\partial t} \right) \frac{P}{\mu(P)} H^2(P) + \int_{p_0}^{P} \left\{ \frac{\partial u_2}{\partial x} u^2 + \frac{\partial v_2}{\partial y} v^2 \right. \\ &+ \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) uv + \frac{p^2}{\mu} \frac{\partial H}{\partial p} \left(u \frac{\partial^2 H_2}{\partial x\, \partial p} + v \frac{\partial^2 H_2}{\partial y\, \partial p} \right) \\ &+ \left. \left[\lambda - \frac{1}{2} \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) \right] \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp. \end{aligned} \tag{12}$$

Since, by assumption, the derivatives of u_1, v_1, H_1 and u_2, v_2, H_2 entering K are continuous, they are bounded in Ω, and by choosing a sufficiently large λ one can obtain the inequality

$$\int_\Omega K\, dx\, dy\, dt \geqslant 0. \tag{13}$$

Let Ω be a truncated cone with bases in the planes $t = 0$ and $t = t_1$; S_0 the lower base of the cone Ω; S_1 the upper base; S_2 the lateral surface of Ω. Suppose that on S_2 the equality

$$\operatorname{tg} nt = \frac{1}{C + V_0}, \tag{14}$$

is satisfied, where n is the outward normal to S_2,

$$v_0 = \max_{S_2} \sqrt{u_1^2 + v_1^2}, \tag{15}$$

$$p_0 \leqslant p \leqslant P$$

$$C = \max\{\sqrt{4\mu_1}, \sqrt{2\mu(P)/\alpha_0}\}. \tag{16}$$

If (u_1, v_1, H_1) and (u_2, v_2, H_2) coincide on S_0, then, taking (13) into account and applying the Ostrogradsky-Gauss formula, instead of (8) we shall have

$$\int_{S_1} Q \, dx \, dy \leqslant -\int_{S_2} (Q \cos nt + R \cos nx + L \cos ny) \, dS. \tag{17}$$

It is verified directly, taking into account (14), (15), and (16), that

$$\int_{S_2} (Q \cos nt + R \cos nx + h \cos ny) \, dS \geqslant 0. \tag{18}$$

From (18), (17), and (9) it follows that $u = v = H = 0$ on S_1.

Thus we have proved

Theorem 2. Problem (1)–(3) has at most one smooth solution.

Let Ω be a right cylinder with bases in the planes $t = 0$ and $t = t_1$; S_0 the lower base of Ω; S_1 the upper base of Ω; S_2 the lateral surface of Ω.

Definition 2. We shall say that a point (x, y, t, p), $(x, y, t) \in S_2$, is an inflow point of the solution (u_1, v_1, H_1) if at this point the inequality

$$u_1 \cos nx + v_2 \cos ny < 0, \tag{19}$$

is satisfied, where $\mathbf{n}$ is the outward normal to S_2.

Theorem 3. If (u_1, v_1, H_1), (u_2, v_2, H_2) are two solutions of problem (1)–(3), coinciding on S_0 and at the inflow points, and $H_1 = H_2$ on S_2, then $(u_1, v_1, H_1) \equiv (u_2, v_2, H_2)$ in Ω.

The author expresses gratitude to Corresponding Member of the Academy of Sciences of the USSR G. I. Marchuk for posing the problem and for guidance.

Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

Received
15 V 1965

Note: Figure translations are in progress. See original paper for figures.