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# RATIONAL POINTS OF A CLASS OF ALGEBRAIC CURVES

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**Abstract**

**Full Text**

UDC 511

*MATHEMATICS*

**V. A. DEM' YANENKO**

## **RATIONAL POINTS OF A CLASS OF ALGEBRAIC CURVES**

*(Presented by Academician I. M. Vinogradov on 4 III 1966)*

The results set forth in the present note concern the problem of finding all rational points of a class of curves of higher genus.

Let  $K$  be a field of algebraic numbers of degree  $n$  over the field of rational numbers. Consider the curve

$$\Gamma : F(x, y) = 0$$

of genus  $> 1$ , admitting over the field  $K$   $r$  linearly independent mappings  $\varphi_i$  ( $i = 1, 2, \dots, r$ ) onto one and the same elliptic curve

$$\Gamma_1 : \Phi(u, v) = 0.$$

The problem is to find all points of the curve  $\Gamma$  rational over the given field  $K$ , under certain restrictions imposed on the curves  $\Gamma$  and  $\Gamma_1$ .

It is known that the mappings of the curve  $\Gamma$  onto the curve  $\Gamma_1$  and the points of the curve  $\Gamma_1$  over  $K$  form groups  $G$  and  $U$  with a finite number of generators. To each point  $a$  of the curve  $\Gamma$  there correspond  $r$  points  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ) of the curve  $\Gamma_1$ .

Two methods are indicated for solving the problem posed.

The first method consists in investigating the points  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ) for linear dependence; in doing so the Tate height of points and the properties of height expounded in the book <sup>(1)</sup> are used in an essential way. In this direction the following results have been obtained.

Let  $g(\varphi)$  be the degree of the mapping  $\varphi$ ; let  $e$  be a function taking only the values  $\pm 1$ ; and let  $H_a$  be the Tate height of the point  $a$ . Then the following theorems hold.

**Theorem 1.** *If*

$$g\left(\sum_{i=1}^r e_i \varphi_i\right) > g\left(\sum_{\substack{i=1 \\ i \neq k}}^r e_i \varphi_i\right) \quad (k = 1, 2, \dots, r)$$

for all values  $e_i$ , then the points  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ),  $a \in \Gamma$ , starting from some  $C$  when  $H_a > C$ , are linearly independent.

Note that in Theorem 1 the mappings  $\varphi_i$  ( $i = 1, 2, \dots, r$ ) are not assumed to be linearly independent; this already follows from the theorem itself.

**Theorem 2.** *If the mappings  $\varphi_i$  ( $i = 1, 2, \dots, r$ ) are linearly independent, then the points of the curve  $\Gamma_1$ ,  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ),  $a \in \Gamma$ , starting from some  $C$  when  $H_a > C$ , are also linearly independent.*

**Theorem 3 (main).** *If the curve  $\Gamma$  admits over the field  $K$   $r$  linearly independent mappings onto one and the same elliptic curve  $\Gamma_1$  of rank  $< r$ , then the curve  $\Gamma$  has a finite number of points rational over this field.*

Note that if the mappings of the curve  $\Gamma$  onto the curve  $\Gamma_1$  are known, then, when the conditions of Theorem 3 are satisfied, the constant  $C$  can be effectively found, and consequently all points of the curve  $\Gamma$  rational over the given field can be found.

Particular cases of the proved theorem are the following examples, considered at the end of the paper <sup>(2)</sup>:

**Example 1.** The curve  $x^4 + y^4 = A$  has a finite number of points rational over the field  $K$ ,  $A \in K$ , if the curve  $u^3 + v^2 = A$  has rank  $\leq 1$  over  $K$ .

**Example 2.** The curve  $x^4 + y^4 = A$  has a finite number of points rational over the field  $K$ ,  $A \in K$ , if the curve  $u^4 + v^2 = A$  has rank  $\leq 1$  over  $K$ .

With respect to the curves considered in Examples 1 and 2, we prove the following additional theorems.

**Theorem 4.** *The curve  $x^6 + y^6 = A$  has a finite number of points rational over the field  $R(1)$ , if the rank of one of the curves  $\Gamma_1 : u^3 + v^2 = A$ ,  $\Gamma_2 : Au^3 - 1 = v^2$ ,  $\Gamma_3 : u^3 + 1 = Av^2$  over  $R(1)$  does not exceed 1, or if the ranks of any two of these curves do not exceed 2.*

**Theorem 5.** *If the rank of the curve  $u^4 + v^2 = A$  over the field  $R(1)$  does not exceed 2, and the rank of the curve  $u^4 + 1 = Av^2 - 1$ , then the curve  $x^4 + y^4 = A$  has a finite number of rational points.*

In the proof of these theorems one uses the fact that the curves  $x^4 + y^4 = A$ ,  $x^6 + y^6 = A$  admit, respectively over the fields  $R(\sqrt[4]{-A})$ ,  $R(\sqrt[6]{-A})$ , a greater number of mappings onto one and the same elliptic curve than over the field  $R(1)$ .

In the second method, one studies the distribution of the points  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ) among the elements of the factor group  $U/2U$ , without using the Tate height. In this method, curves of genus  $> 1$  for which the order of the automorphism group is equal to 2 are considered in greater detail. The following theorems are proved.

**Theorem 6.** *If the rank of one of the curves  $\Gamma_1, \Gamma_2, \Gamma_3$  over the field  $R(\varepsilon)$ ,  $\varepsilon^2 + \varepsilon + 1 = 0$ , does not exceed 2, then the curve  $x^6 + y^6 = A$  has no points in this field, except for the cases:  $A = 1$ ,  $(x, y) = (\varepsilon_1, 0), (0, \varepsilon_2)$ ;  $A = 2$ ,  $(x, y) = (\varepsilon_1, \varepsilon_2)$ ,  $\varepsilon_1^6 = \varepsilon_2^6 = 1$ .*

**Theorem 7.** *If the rank of the curve  $u^4 + v^2 = A$  over the field  $R(\sqrt{-1})$  does not exceed 2, then the curve  $x^4 + y^4 = A$  has no points in this field, except for the cases:  $A = 1$ ,  $(x, y) = (\varepsilon_1, 0), (0, \varepsilon_2)$ ;  $A = 2$ ,  $(x, y) = (\varepsilon_1, \varepsilon_2)$ ;  $\varepsilon_1^4 = \varepsilon_2^4 = 1$ .*

**Theorem 8.** *If the rank of the curve  $(A^2 - 4)u^4 + 2A(A + 2)u^2 + A^2 - 4 = v^2$  over the field  $R(1)$  does not exceed 2, then the curve  $x^4 + y^4 + 1 = A(x^2y^2 + x^2 + y^2)$  has no rational points, except for the cases:  $A = (2a^4 + 1)/(a^4 + 2a^2)$ ,  $(x, y) = (\pm a, \pm a)$ ;  $A = (a^4 + 2)/(2a^2 + 1)$ ,  $(x, y) = (\pm a, \pm 1), (\pm 1, \pm a)$ .*

In the proof of Theorems 6, 7, 8 it is established: first, that none of the points  $\varphi_i(a)$  ( $i = 1, 2, \dots, r$ ) of the corresponding elliptic curve can fall into the class  $2U$ ; second, that no two points among  $\varphi_i(a)$  can fall into one and the same class of the factor group  $U/2U$ .

Under the condition that  $K = R(1)$ , we put forward the following conjecture for the curves of Theorems 6, 7, 8: if the points  $\varphi_i(a), \varphi_i(b)$  ( $i = 1, 2, \dots, r$ ),  $r \geq 2$ , are distributed entirely in two classes of the factor group  $U/2U$ , then  $a = b$ . From the validity of this hypothesis the following results would follow:

1. The curves  $x^4 + y^4 = A$ ,  $x^4 + y^4 + 1 = A(x^2y^2 + x^2 + y^2)$  have in the field  $R(1)$ , respectively, no more than  $2^{r_1-1}(2^{r_1} - 1)$ ,  $2^{r_2-1}(2^{r_2} - 1)$  points, where  $r_1, r_2$  are the ranks of the curves  $u^4 + v^2 = A$ ,  $(A^2 - 4)u^4 + 2A(A + 2)u^2 + A^2 - 4 = v^2$  over this field.
2. The curve  $x^6 + y^6 = A$  has in the field  $R(1)$  no more than  $2^{s-1}(2^s - 1)$  points, where  $s = \min(r_1, r_2, r_3)$ , and  $r_1, r_2, r_3$  are the ranks of the curves  $u^3 + v^2 = A$ ,  $Au^3 - 1 = v^2$ ,  $u^3 + 1 = Av^2$  over this field.

Steklov Mathematical Institute  
Academy of Sciences of the USSR

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*Note: Figure translations are in progress. See original paper for figures.*

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