ASYMPTOTICS OF THE SPECTRAL FUNCTION OF A SINGULAR DIFFERENTIAL OPERATOR OF ORDER \(2m\)

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Abstract

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ASYMPTOTICS OF THE SPECTRAL FUNCTION OF A SINGULAR DIFFERENTIAL OPERATOR OF ORDER $2m$

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Let us consider a formally symmetric differential expression $l(y)$, defined on the entire line or half-line, of the form

$$l(y) = (-1)^m y^{(2m)} + p_{2m-2}(x) y^{(2m-2)} + ... + p_0(x) y.$$

Let L_0 be the minimal operator generated by $l(y)$. We shall assume that L_0 has equal defect indices. Denote by L an arbitrary self-adjoint extension of it. It is known (see, for example, ([1])) that the spectral family E_λ of the operator L is a family of integral operators with kernel $\theta(x, \xi, \lambda)$. This kernel is called the spectral function.

Of great interest is the study of the asymptotic behavior of $\theta(x, \xi, \lambda)$ for large λ. For the Sturm-Liouville operator this question was completely solved in the well-known works of B. M. Levitan ([2]) and V. A. Marchenko ([3]). They proved that, as $\lambda \to \infty$, uniformly in every finite domain of variation of x, ξ, the equality

$$\theta(x, \xi, \lambda) = \theta_0(x, \xi, \lambda) + o(1),$$

holds, where $\theta_0(x, \xi, \lambda)$ is the spectral function of the operator $-y''$.

The method used by B. M. Levitan and V. A. Marchenko is connected with consideration of the corresponding wave equation and the subsequent use of a Tauberian theorem for the Fourier transform, which was found by the named authors specially for obtaining the asymptotics of $\theta(x, \xi, \lambda)$ by solving the wave equation. Their method does not extend to operators of order higher than 2, since with such an operator it is not possible to associate a hyperbolic equation (hyperbolicity makes it comparatively simple to investigate the Cauchy problem for arbitrary growth of the coefficients).

It should be said that L. Gårding ([4]), studying the kernel of the resolvent of the operator L, found the asymptotics of the spectral function $\theta(x, \xi, \lambda)$ of an arbitrary semibounded elliptic operator, and hence also of our operator (at least in the case of positive L_0). However, his estimate of the remainder term is quite rough, $o(\lambda^{1/2m})$, whereas for the Sturm–Liouville equation the remainder has the form $o(1)$, which is very significant for various applications. For operators with constant coefficients L. Gårding obtained in ([4]) a very precise estimate. A certain attempt to improve the estimate of the remainder for elliptic operators with variable coefficients was made in the work of T. Bergendal ([5]).

We use, for solving the problem posed, the corresponding parabolic equation

$$\partial u/\partial t = -Lu. \tag{1}$$

For elliptic operators with constant coefficients it was successfully used by L. Gårding ([4]). It was first applied by Minakshisundaram to the Laplace operator in a finite domain.

Theorem 1. Let the operator L_0, defined on the whole axis, be semibounded. Its coefficients are such that $p_{2m-2}(x)$ is a piecewise smooth function, and the remaining coefficients are locally summable. Then in every finite domain (x, ξ), as $\lambda \to \infty$, for the spectral function $\theta(x, \xi, \lambda)$ of any semibounded self-adjoint extension, one has

$$\theta(x, \xi, \lambda) = \theta_0(x, \xi, \lambda) + o(1), \tag{2}$$

where

$$\theta_0(x, \xi, \lambda) = \frac{1}{\pi} \frac{\sin \lambda^{1/2m}(x-\xi)}{x-\xi}$$

is the spectral function of the operator $(-1)^m y^{(2m)}$.

If the operator is given on the half-axis with self-adjoint boundary conditions

$$B_j(y) = \sum_{i=1}^{2m} b_{ij} y^{i-1}(0),$$

then a theorem analogous to the one formulated above holds, provided the boundary conditions satisfy a certain requirement. Namely, let λ_i^+ be the roots of the equation $\tau^{2m} - \lambda = 0$ lying in the upper half-plane, and

$$M^+(\tau, \lambda) = \prod_{i=1}^{m} (\tau - \lambda_i^+).$$

Denote by $B_j'(\tau, \lambda)$ the remainder upon division of

$$B_j(\tau) = \sum_{i=1}^{2m} b_{ij}\tau^{i-1}$$

by $M^+(\tau, \lambda)$. We require that:

1°. det $\|b'_{ij}\| \neq 0$ (the solvability condition,

$$B'_j(\tau, \lambda) = \sum_{i=1}^{m} b'_{ij}\tau^{i-1}$$

).

2°. The elements c_{ji} of the matrix inverse to the matrix $\|b'_{ij}\|$ satisfy the inequality

$$|c_{ji}| \leq C|\lambda_i^+|^{-r_j+i-1},$$

where r_j is the highest order of differentiation in the j-th boundary condition B_j.*

Theorem 2. Let L_0 be the minimal operator generated by a formally symmetric semibounded differential expression, given on the half-axis $0 \leq x < \infty$. The boundary conditions satisfy conditions $1°, 2°$. The coefficients $p_i(x)$ are the same as in Theorem 1. Then formula (2) holds, where $\theta(x, \xi, \lambda)$ is the spectral function of any semibounded self-adjoint extension, and $\theta_0(x, \xi, \lambda)$ is the spectral function of the operator $(-1)^m y^{(2m)}$ on the half-axis with the same boundary conditions B_j.

The proof of Theorem 1 proceeds according to the following scheme. First we investigate the spectral function $\tilde{\theta}_0(x, \xi, \lambda)$ of the operator $\widetilde{L}$, whose coefficients are finite, i.e., vanish outside some interval. For this it is necessary to act directly. Namely, consider the equation for the kernel $\widetilde{K}(x, \xi, \lambda)$ of the resolvent of the operator $\widetilde{L}$:

$$\widetilde{K}(x, \xi, \lambda) = K_0(x, \xi, \lambda) + \int_{-\infty}^{\infty} K_0(x, s, \lambda)\, \widetilde{P}\left(s, \frac{d}{ds}\right) \widetilde{K}(s, \xi, \lambda)\, ds, \qquad (3)$$

where $K_0(x, \xi, \lambda)$ is the kernel of the resolvent of the operator $(-1)^m y^{(2m)}$, and $\widetilde{P}(x, d/dx)$ is the operator containing all lower coefficients of the operator $\widetilde{L}$ (its coefficients are finite). It can be shown that for large λ equation (3) can be solved by the method of successive approximations and, under the conditions imposed on the coefficients $p_i(x)$, uniformly in x, ξ on the whole line, as $\lambda \to \infty$,

$$\widetilde{K}(x, \xi, \lambda) = K_0(x, \xi, \lambda) + o(1)/|\lambda|. \qquad (4)$$

* It is possible that this condition is a consequence of 1°.

To obtain (4) it is necessary only to pay attention to the first iteration

$$K_1(x,\xi,\lambda) = \int_{-\infty}^{\infty} K_0(x,s,\lambda)\,\widetilde{P}\left(s,\frac{d}{ds}\right) K_0(s,\xi,\lambda)\,ds,$$

since the subsequent iterations have, with respect to λ, order of decrease

$$|\lambda|^{-(2m-1)/m}.$$

Integrating the left- and right-hand sides of (4) over a closed contour Γ_η in the complex plane $\lambda = \eta + i\tau$, which passes through the point η on the real axis, we obtain, as $\eta \to \infty$,

$$\tilde{\theta}_0(x,\xi,\eta) = \theta_0(x,\xi,\eta) + o(1), \tag{5}$$

where $\theta_0(x,\xi,\eta)$ is the spectral function of the operator $(-1)^m y^{(2m)}$.

Let us now suppose that we have to establish formula (2) in the domain $S(a \le x \le b,\ a_1 \le \xi \le b_1)$. Take the domain $U(a-h \le x \le b+h)$ and consider the "truncated" operator whose coefficients coincide with the coefficients $p_i(x)$ of the operator L in the domain U, and outside $U_\varepsilon(a-h-\varepsilon \le x \le b+h+\varepsilon)$ vanish.

Denote by $\widetilde{G}_0(x,\xi,t)$ the Green' s function of the Cauchy problem for the parabolic equation

$$\partial u/\partial t = -\widetilde{L}u.$$

If $G(x,\xi,t)$ is the kernel of the integral operator e^{-tL}, i.e. the Green' s function of the Cauchy problem for the operator equation (1), then, as can be verified with the aid of Green' s formula, in the domain $S(a \le x \le b,\ a_1 \le \xi \le b_1)$ the equality

$$G(x,\xi,t) - \widetilde{G}_0(x,\xi,t) = O(1)e^{-ch/t^{1/(2m-1)}}, \tag{6}$$

holds, where c is a constant independent of ε, h, t.

Since $G(x,\xi,t)$ is the Laplace-Stieltjes transform of $\theta(x,\xi,\lambda)$, it follows from (6) that

$$\int_0^{\infty} e^{-\lambda t}\,d\theta(x,\xi,\lambda) = \int_0^{\infty} e^{-\lambda t}\,d\tilde{\theta}_0(x,\xi,\lambda) + O(1)e^{-ch/t^{1/(2m-1)}}, \tag{7}$$

where $\tilde{\theta}_0(x,\xi,\lambda)$ is the spectral function of the "truncated" operator $\widetilde{L}$. After multiplying the left- and right-hand sides of (7) by $e^{-\mu t}$ and integrating with respect to t from 0 to ∞, we obtain

$$\int_0^{\infty} \frac{d\theta(x,x,\lambda)}{\lambda+\mu} = \int_0^{\infty} \frac{d\tilde{\theta}_0(x,x,\lambda)}{\lambda+\mu} + O(1)e^{-c^* h^* \mu^{1/2m}}, \tag{8}$$

where c^* is a constant depending only on c, $h^* = h^{(2m-1)/2m}$. Next, multiply the left- and right-hand sides of (8) by $K(s,\mu)$

$$K(s,\mu) = \frac{1}{\pi} e^{-s\mu^{1/2m}\cos\pi/2m} \left\{ \sin\left(s\mu^{1/2m} \sin\frac{\pi}{2m} \right) \right\}$$

and again integrate with respect to μ from 0 to ∞. It turns out that

$$\int_0^\infty \frac{K(s,\mu)}{\lambda+\mu}\,d\mu = e^{-s\lambda^{1/2m}}.$$

Therefore we have

$$\int_0^\infty e^{-s\lambda^{1/2m}}\,d\theta(x,x,\lambda) = \int_0^\infty e^{-s\lambda^{1/2m}}\,d\tilde{\theta}_0(x,x,\lambda) + G(s),$$

or

$$\int_0^\infty e^{-s\lambda}\,d\rho(\lambda) = \int_0^\infty e^{-s\lambda}\,d\sigma(\lambda) + G(s),$$

where

$$G(s) = \int_0^\infty K(s,\mu)O(1)e^{-c_1 h^* \mu^{1/2m}}, \qquad \rho(\lambda) = \theta(x,x,\lambda^{2m}), \quad \sigma(\lambda) = \tilde{\theta}_0(x,x,\lambda^{2m}).$$

The functions $\rho(\lambda)$ and $\sigma(\lambda)$ are nondecreasing. From formula (5) it is seen that

$$\sigma(\lambda) = \tilde{\theta}_0(x,x,\lambda^{2m}) = \frac{1}{\pi}\lambda + o(1),$$

whence it follows that

$$\lim_{\lambda\to\infty} \operatorname*{Var}_{\lambda}^{\lambda+1/a} \sigma(\xi) = \frac{1}{\pi a}.$$

It is also clear from the form of $K(s,\mu)$ that the function $G(s)$ is an analytic function of $s = \eta + it$ in the strip $|t| < \tilde{h}$, $\tilde{h} = h^* c^* / \cos\frac{\pi}{2m}$.

Now the Tauberian theorem of V. A. Marchenko ([3]) gives

$$\varlimsup_{\lambda\to\infty} \left| \frac{\rho(\lambda+0)+\rho(\lambda)}{2} - \sigma(\lambda) - G(0) \right| \le A \operatorname*{Var}_{\lambda}^{\lambda+1/\tilde{h}} \sigma(\xi) = \frac{A}{\tilde{h}\pi}, \tag{9}$$

where A is an absolute constant.

It turns out that, in our case, $G(0) = 0$. This fact follows from the fact that (as follows from (9)) $\theta(x,x,\lambda) - \tilde{\theta}_0(x,x,\lambda) = \psi(x,\lambda) = O(1)$, and from the general Tauberian theorem of N. Wiener ([6]). Now the assertion of Theorem 1 in the case $x = \xi$ follows already from (5) and (9). After this one may consider the function

$$\Phi(x,\xi,\lambda) = \theta(x,x,\lambda) + 2\theta(x,\xi,\lambda) + \theta(\xi,\xi,\lambda),$$

which will be nondecreasing.

The proof of Theorem 2 is technically somewhat more complicated, but is carried out according to exactly the same scheme. In the same way one can obtain asymptotics for the derivatives of the spectral function. Let us further note that from the results of N. Nilsson ([7]) on the estimate of the spectral function on the negative part of the spectrum it follows that Theorems 1 and 2 are valid, at least, for sufficiently smooth coefficients and for non-semi-bounded operators. From the theorems proved there follow theorems on the equiconvergence of expansions. Theorems 1 and 2 can also be successfully applied to the computation of traces for singular operators of high order according to the scheme, for example, of the work ([8]).

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References

[1] L. Gårding, C. R. II Congrès Math. Scandinaves, Lund, 1953, p. 44.
[2] B. M. Levitan, Izv. AN SSSR, ser. matem., 19, 33 (1955).
[3] V. A. Marchenko, Izv. AN SSSR, ser. matem., 19, 38 (1955).
[4] L. Gårding, Kungl. fysiogr. sällkap. Lund Förhandl., 24, No. 21 (1954).
[5] G. Berndal, Thesis, Lund, 1959.
[6] G. Hardy, Divergent Series, IL, 1951, p. 366.
[7] N. Nilsson, Math. Scand., 9, 107 (1961).
[8] M. G. Gasymov, DAN, 150, No. 6 (1962).

Note: Figure translations are in progress. See original paper for figures.