ON THE VALUES TAKEN BY ENTIRE POLYANALYTIC FUNCTIONS

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Abstract

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ON THE VALUES TAKEN BY ENTIRE POLY-ANALYTIC FUNCTIONS

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1. A function $f(z)$ is called an **entire polyanalytic** function (of order $n+1$) if, in the whole plane of the complex variable z, it admits a representation of the form

$$f(z) = \sum_{k=0}^{n} \bar{z}^k \cdot f_k(z), \tag{1}$$

where $f_k(z)$ $(k = 0, 1, \dots, n)$ are entire analytic functions, and $\bar{z}$ is the variable conjugate to the variable z; in other words, $f(z)$ is a polynomial in $\bar{z}$ with coefficients that are entire functions of z. An example of an entire polyanalytic function may be a polyanalytic polynomial, i.e., a polynomial $P(z, \bar{z})$ in the two conjugate variables z and $\bar{z}$. Everywhere below, for simplicity, we shall assume that $f_n(z) \not\equiv 0$ and $f_n(0) \neq 0$. This does not affect the generality of the results presented below.

In this note we intend to clarify to what extent the Picard theorem, known for entire analytic functions, is preserved for functions of the form (1).

2. We first consider **polyanalytic polynomials**. Among them there are, obviously, those that omit infinitely many complex values (for example, $z^3 + \bar{z}^3 + i$). However, the following assertion is true:

Theorem 1 (the fundamental theorem of algebra for polyanalytic polynomials). *If the exact degree of a polyanalytic polynomial with respect to the pair of variables $z, \bar{z}$ is more than twice the exact degree of this polynomial with respect to one of the variables $z, \bar{z}$, then the polynomial has at least one root.*

Indeed, let the polynomial

$$P(z, \bar{z}) \equiv \sum_{k=0}^{n} \bar{z}^k \cdot P_k(z), \quad \text{where} \quad P_k(z) = \sum_{\nu=0}^{s-k} a_\nu^{(k)} z^\nu,$$

have, with respect to the pair of variables $z, \bar{z}$, exact degree s, so that not all the numbers $a_{s-k}^{(k)}$ are equal to zero. From the assumption that $P(z, \bar{z})$ has no zeros, one can deduce that, for any $t > 0$, the analytic polynomial (in z)

$$\Phi(z;t) = \sum_{k=0}^{n} t^{2k} z^{n-k} P_k(z)$$

has exactly n zeros in $|z| < t$, while the polynomial

$$\Psi(z;t) = \Phi(tz;t)/t^{s+n}$$

has exactly n zeros in $|z| < 1$. But for $|z| < 1$ and $t \to \infty$,

$$\Psi(z;t) = \left(a_s^{(0)} z^{s+n} + a_{s-1}^{(1)} z^{s+n-2} + \cdots + a_{s-n}^{(n)} z^{s-n}\right) + o(1).$$

By Hurwitz' s theorem it follows from this that, for large t, $\Psi(z;t)$ has in $|z| < 1$ no fewer than $s - n$ zeros. Thus, $n \geq s - n$, $s \leq 2n$. If

$P(z, \bar{z})$ has degree n_1 with respect to z, then $\overline{P(z, \bar{z})}$ has degree n_1 with respect to $\bar{z}$, and therefore $s \leqslant 2n_1$. The theorem is proved.

Consequence. *A polyanalytic polynomial satisfying the condition of Theorem 1 assumes every complex value.*

Let us additionally note that for any integers n and s for which $n \leqslant s \leqslant 2n$, there exists a polyanalytic polynomial of order $n + 1$ (i.e. of the form (1)) and of exact degree s, omitting a set of values of infinite planar measure. Such, for example, is the polynomial $(z + \bar{z})^{2n-s} \cdot (z \cdot \bar{z})^{s-n}$, which assumes only real values.

3. We now turn to **entire transcendental polyanalytic functions**. Invoking the A. Cartan formula known in the theory of value distribution ((1), p. 179), one can establish that for such functions the well-known Sokhotski–Weierstrass theorem remains valid. However, by other means one can obtain a stronger result:

Theorem 2 (Picard' s big theorem for entire polyanalytic functions). *Every entire transcendental polyanalytic function assumes every complex value, with the possible exception of one, on an unbounded set of points.*

Indeed, suppose the function (1) assumes two values (for definiteness, 0 and 1) only on a bounded set M; we shall show that $f(z)$ is a polynomial.

There exists an r_0 such that M lies in $|z| < r_0$. Hence there exist constants p_1 and q_1 such that for all $t \geqslant r_0$ and $\lambda > 2$ on the circle $\Gamma\{|z - t| = \lambda t\}$

$$Bp_\Gamma f(z) \equiv \frac{1}{2\pi} \int_\Gamma d \arg f(z) = p_1, \qquad Bp_\Gamma[f(z) - 1] = q_1.$$

On Γ the function $f(z)$ coincides with the analytic function

$$\Phi(z;\lambda t)=\sum_{\nu=0}^{n}\lambda^{2\nu}t^{2\nu}\frac{\varphi_\nu(z;t)}{(z-t)^\nu},$$

where

$$\varphi_\nu(z)=\sum_{k=\nu}^{n}C_k^\nu f_k(z)t^{k-\nu}\quad(\nu=0,1,\dots,n).$$

Therefore $Bp_\Gamma\Phi(z;\lambda t)=p_1,\ Bp_\Gamma[\Phi(z;\lambda t)-1]=q_1$. Hence it is seen that $\Phi(z;\lambda t)$ assumes the values 0 and 1 inside Γ respectively p and q times, where $p=p_1+n,\ q=q_1+n$, while the function

$$\varphi(z)\equiv\Phi(tz;\lambda t)\tag{2}$$

assumes the same values in $\delta\{|z|<1\}$, respectively, no more than p and q times. Put $\lambda=t^s$, where $s=2n+p$, and

$$\varphi(z)=\sum_{k=0}^{\infty}a_k z^k.\tag{3}$$

As $t\to\infty$,

$$a_k=o(\lambda^{3n})\quad(k=0,1,\dots,p).\tag{4}$$

V. Zakser, generalizing the well-known Schottky theorem ((2), p. 215), proved that if some function of the form (3) is regular in $\delta\{|z|<1\}$ and there assumes the values 0 and 1 no more than p and q times respectively, and the coefficients $a_0,\dots,a_p$ satisfy the inequalities $|a_\nu|<K_\nu$ ($\nu=0,\dots,p;\ K_0>e$), then in the disk $|z|\leqslant 1-\theta$ ($0<\theta<1$) the inequality

$$|\varphi(z)|<(K_0+K_1+\dots+K_p)^{C/\theta},$$

holds, where C is a constant depending neither on θ, nor on $\varphi(z)$, nor on the numbers $K_0,\dots,K_p$. Taking $\theta=1/2$ and taking (4) into account, one can show that in our case there must exist for (3) a number m such that in $|z|\leqslant 1/2$ for as $t\to\infty$, $\varphi(z)=o(t^m)$. Consequently, for $\psi(z)\equiv(z-1)^n\varphi(z)$ we also have the estimate (for $|z_1|\leqslant 1/2$) $\psi(z)=o(t^m)$.

Let

$$f_\nu(z) = \sum_{\mu=0}^{\infty} c_\mu^{(\nu)} z^\mu \quad (\nu = 0, 1, \dots, n).$$

Applying Cauchy' s inequalities to the function $\psi(z)$, we obtain that, as $t \to \infty$ and for every μ,

$$\lambda^{2n} t^{n+\mu} \left| o(1) + c_\mu^{(n)} \right| = o(t^m),$$

whence, for all sufficiently large μ, $c_\mu^{(n)} = 0$, i.e. $f_n(z)$ is a polynomial. By an analogous argument one can then successively show that the remaining functions $f_\nu(z)$ $(\nu = n-1, \dots, 0)$ are polynomials, so that Theorem 2 is true.

We note that for $n = 1$ Theorem 2 was proved by another method in ([3]). From Theorems 2 and 1 there follows

Corollary (the little Picard theorem for entire polyanalytic functions). *Every entire polyanalytic function of order $n+1$ which is not, as a function of the pair of variables $z, \bar{z}$, a polynomial of degree $2n$ (or of smaller degree), assumes every complex value, with at most one possible exception.*

Refinements of the last assertion are known if $n = 1$ and the function (1) satisfies certain additional conditions (see ([4,5])).

We also note that for the so-called **reduced** entire polyanalytic functions (i.e. functions of the form (1) satisfying, for $k = 0, 1, \dots, n$, the condition $f_k(z) = z^k \varphi_k(z)$, where $\varphi_k(z)$ are entire functions) the little Picard theorem can be obtained considerably more simply (without referring to Theorems 1 and 2), if one uses the known estimates of A. Pfluger ([6]) or L. Ahlfors ([7]) for the classical Schottky theorem.

From Theorem 2 there follows the following generalization of the well-known Picard theorem for entire **analytic** functions:

Let $P(x, y)$ and $Q(x, y)$ be arbitrary polynomials in the real variables x, y, with $P(x, y) \not\equiv 0$; let $f(z)$ be an entire analytic function of the complex variable $z = x + iy$. If the function

$$P(x, y) \cdot f(z) + Q(x, y) \tag{5}$$

assumes each of two complex values a and b $(a \neq b)$ on a bounded set of points, then $f(z)$ is a polynomial.

For the proof it suffices to observe that the function (5) is entire polyanalytic.

4. For some classes of functions of the form (1), Theorem 2 can be strengthened. In particular, the following assertion is true:

If in the representation (1) of an entire polyanalytic function $f(z)$, with $n \geqslant 1$, the function $f_n(z)$ has infinitely many zeros but is not identically zero, then $f(z)$ assumes every complex value a, without exception, on an unbounded set of points.

The proof can be obtained by the method applied in ([3]) (see Lemma 1).

5. Theorem 2 and its corollary do not remain valid for the whole class of transcendental meromorphic polyanalytic functions (i.e. nonrational functions of the form (1), where all $f_k(z)$ are meromorphic), since there exist such transcendental meromorphic functions of the form (1) which omit a set of values of infinite planar measure. This is the case, for example, if the meromorphic function (1) is representable in the form

$$\sum_{\nu=1}^{\infty} A_\nu \cdot \left(P_\nu(z)/\overline{P_\nu(z)}\right),$$

where

$$\sum_{\nu=1}^{\infty} |A_\nu| < \infty,$$

and $P_\nu(z)$ $(\nu = 1, 2, \ldots)$ are polynomials in z of degree n (or

below), and the union of the sets of their zeros has the single limit point ∞.

However, under certain restrictions on the set of poles of the functions $f_k(z)$ $(k = 0, \ldots, n)$ occurring in (1), it is possible to obtain, also for meromorphic polyanalytic functions, analogues of Picard' s theorem. We give an example of a proposition of this type (restricting ourselves to the case $n = 1$):

Let $f(z) = f_0(z) + \bar{z} f_1(z)$ be a transcendental meromorphic polyanalytic function (of order 2), with $f_1(z) \not\equiv 0$, and suppose that the number of common poles of the meromorphic functions $f_0(z)$ and $f_1(z)$ is at most finite; then $f(z)$ assumes every complex value, except perhaps one, on an unrestricted set of points.

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Note: Figure translations are in progress. See original paper for figures.