SOME THEOREMS ON POLIVECTORS

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SOME THEOREMS ON POLIVECTORS

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We shall call a k-vector $^{(1)}$ P a **k-vector of simple type** if it can be represented in the form

$$P = \sum_{i=1}^{r} [\alpha_{i1}\alpha_{i2} \dots \alpha_{ik}], \tag{1}$$

where all kr covariant vectors α_{ij} are linearly independent.

Theorem 1. For $k > 2$, the simple polivectors $[\alpha_{i1}\alpha_{i2} \dots \alpha_{ik}]$ are determined uniquely (up to the order in which they occur on the right-hand side of (1)).*

Theorem 2. A polivector P is a k-vector of simple type $(k > 4)$ if and only if, for every vector p, the tensor P_p, defined by the equality

$$P_p x_1 x_2 \dots x_{k-1} \equiv P p x_1 x_2 \dots x_{k-1},$$

is a $(k-1)$-vector of simple type whose rank-to-rank ratio for P does not exceed $(k-1)/k$.

Let $K(T_1, T_2, \dots, T_n)$ be some concomitant of the tensors $T_1, T_2, \dots, T_n$. A **general solution** of the equation $K(T_1, T_2, \dots, T_n) = 0$ will mean n concomitants $L_i(V_1, V_2, \dots, V_m)$ $(i = 1, 2, \dots, n)$ of the tensors $V_1, V_2, \dots, V_m$, having the following properties: a) the equality $K(L_1(V_1, V_2, \dots, V_m), L_2(V_1, V_2, \dots, V_m), \dots, L_n(V_1, V_2, \dots, V_m)) = 0$ is an identity for arbitrary tensors $V_1, V_2, \dots, V_m$ of the corresponding valencies; b) if some $\hat{T}_1, \hat{T}_2, \dots, \hat{T}_n$ constitute a solution of the equation, then there exist tensors $\hat{V}_1, \hat{V}_2, \dots, \hat{V}_m$ such that $\hat{T}_i = L_i(\hat{V}_1, \hat{V}_2, \dots, \hat{V}_m)$.

Let P be the polivector (1). Introduce the notation

$$P_{(j)} \equiv \sum_{i=1}^{j} [\alpha_{i1}\alpha_{i2} \dots \alpha_{ik}] \quad (j = 1, 2, \dots, r).$$

Theorem 3. For odd k, the general solution of the equation $[P_{(r)}T] = 0$ (T an unknown l-vector) has the form

$$T = [P_{(r)}U] + \sum_{i=1}^{k} [\alpha_{ri} V_i],$$

where U is an arbitrary $(l-k)$-vector, and V_i $(i = 1, 2, \dots, k)$ is the general solution of the equation $[P_{(r-1)}V] = 0$.

Theorem 4. For even k, the general solution of the equation $[P^m_{(r)}T] = 0$ (T an unknown l-vector) has the form

$$X = -\frac{1}{m}[B_{(r-1)}U] + [\alpha_{r1}\alpha_{r2} \dots \alpha_{rk}U] + \sum_{i=1}^{k} [\alpha_{ri} V_i] + W,$$

where U is the general solution of the equation $[B^{m+1}_{(r-1)}U] = 0$; V_i is the general solution of the equation $[B^{m}_{(r-1)}V] = 0$; W is the general solution of the equation $[B^{m-1}_{(r-1)}W] = 0$.

* It is known that for $k = 2$ the covariant vectors α_{ij} and β_{ij} occurring in two representations of a bivector P in the form (1) are connected by linear dependences whose coefficients form a symplectic matrix.

Theorem 5. The general solution of the equation

$$\sum_{\substack{i_1, i_2, \dots, i_r = 1 \\ i_1 < i_2 < \dots < i_r}}^{n} [\alpha_{i_1}\alpha_{i_2} \dots \alpha_{i_r} T_{i_1 i_2 \dots i_r}] = 0,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are given linearly independent covariant vectors, and $T_{i_1 i_2 \dots i_r}$ are unknown k-vectors, has the form

$$T_{i_1 i_2 \dots i_r} = \sum_{l=1}^{n} [\alpha_l V_{l i_1 i_2 \dots i_r}],$$

where the $(k-1)$-vectors $V_{l i_1 i_2 \dots i_r}$ are connected by the relations

$$\sum_{(l, i_1, i_2, \dots, i_r)} (\pm V_{l i_1 i_2 \dots i_r}) = 0 \quad (l \neq i_t)$$

(the summation is over permutations of the indices $l, i_1, i_2, \dots, i_r$; the plus or minus sign is taken according to the parity or oddness of the corresponding substitution; $V_{l i_1 i_2 \dots i_r} = 0$ if the inequality $i_1 < i_2 < \dots < i_r$ is not satisfied).

Theorem 6. The general solution of the equation

$$\sum_{i=1}^{n}(\alpha_{i1}\alpha_{i2}\ldots\alpha_{ir}T_i)=0,$$

where α_{ij} are nr given linearly independent covariant vectors, and $T_1, T_2, \ldots, T_n$ are unknown k-vectors, has the form

$$T_i=\sum_{\substack{j=1\\ j\neq i}}^{n}[\alpha_{j1}\alpha_{j2}\ldots\alpha_{jr}V_{ij}]+\sum_{l=1}^{r}[\alpha_{il}W_{il}],$$

where W_{il} are arbitrary $(k-1)$-vectors, and V_{ij} are $(k-r)$-vectors connected by the relations $V_{ij}=(-1)^{r+1}V_{ji}$.

Each of Theorems 5 and 6 generalizes the well-known lemma of É. Cartan (2), which is obtained if in these theorems one puts $r=1$.

The proofs of Theorems 1–6 are based on the following lemmas.

Lemma 1. Let A be a k-vector and α a covariant vector. The propositions: a) there exists a $(k-1)$-vector B such that $A=[\alpha B]$; b) $[\alpha A]=0$; c) if $\alpha x_1=\alpha x_2=\cdots=\alpha x_k=0$, then $Ax_1x_2\ldots x_k=0$, are equivalent.

Lemma 2. Let $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1s_1}; \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2s_2}, \ldots, \alpha_{r1}, \alpha_{r2}, \ldots, \alpha_{rs_r}$ be $s_1+s_2+\cdots+s_r$ linearly independent covariant vectors, and let A be a k-vector. If, whatever the set of numbers $i_1, i_2, \ldots, i_r$ $(i_l\leq s_l)$, one has $Ax_1x_2\ldots x_k=0$ when $\alpha_{1i_1}x_i=\alpha_{2i_2}x_i=\cdots=\alpha_{ri_r}x_i=0$ for all $i=1,2,\ldots,k$, then there exist polyvectors $B_1, B_2, \ldots, B_r$ of valencies respectively $k-s_1, k-s_2, \ldots, k-s_r$ such that

$$A=[\alpha_{11}\alpha_{12}\ldots\alpha_{1s_1}B_1]+[\alpha_{21}\alpha_{22}\ldots\alpha_{2s_2}B_2]+\cdots+[\alpha_{r1}\alpha_{r2}\ldots\alpha_{rs_r}B_r].$$

These same lemmas make it possible to prove a further series of propositions on polyvectors.

Theorem 7. In order that a simple k-vector A and a simple l-vector B have exactly s linearly independent common divisors (1), it is necessary and sufficient that the relations

$$AB_{[l+k-s+1]}=0, \qquad AB_{[l+k-s]}\neq 0$$

hold (the symbol $[m]$ written below means that the tensor AB has been alternated with respect to arbitrary m arguments).

Theorem 8. If a simple k-vector A and a simple l-vector B have exactly s linearly independent common divisors, then the tensor

$$AB_{[l+k-s]}$$

is the product of two simple polyvectors, one of which is divisible by all common divisors of the polyvectors A and B, and the other by every covariant vector by which either A or B is divisible (this assertion is analogous to the theorem of elementary number theory stating that the product of the greatest common divisor and the least common multiple of two natural numbers is equal to the product of these numbers).

Theorems 7 and 8 give a new method for calculating the invariants h and s of a trivector, introduced by G. B. Gurevich (3), and also lead naturally to new invariants of polyvectors.

The following proposition strengthens A. M. Lopshits’ s theorem (4).

Theorem 9. *In order that three simple bivectors A, B, C, no two of which are proportional, have a common divisor, it is necessary and sufficient that the relation $Ax\,y\,Bz\,u\,Cv\,w = 0$ hold.*

One of the simplest polyvectors not belonging to the class of polyvectors of simple type is the trivector D of type (6.3) (3), whose canonical form is

$$D = [\alpha_1\beta_2\beta_3] + [\alpha_2\beta_3\beta_1] + [\alpha_3\beta_1\beta_2],$$

where all six covariant vectors appearing on the right-hand side are linearly independent. We give an analogue of Theorem 1 for the trivector D.

Theorem 10. *If*

$$D = [\alpha_1\beta_2\beta_3] + [\alpha_2\beta_3\beta_1] + [\alpha_3\beta_1\beta_2]$$

and

$$D = [\tilde{\alpha}_1\tilde{\beta}_2\tilde{\beta}_3] + [\tilde{\alpha}_2\tilde{\beta}_3\tilde{\beta}_1] + [\tilde{\alpha}_3\tilde{\beta}_1\tilde{\beta}_2]$$

are two representations of the trivector D in canonical form, then $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are linear combinations of α_i and β_i with coefficients forming a sixth-order matrix of the form

$$\left\| \begin{matrix} \dfrac{M}{\det M} & 0 \\ N & M \end{matrix} \right\|, \tag{2}$$

where M and N are third-order matrices, and

$$\det M \neq 0, \qquad \operatorname{sp} NM^{-1} = 0. \tag{3}$$

From the very formulation of the problem that led to matrices of the form (2)–(3), it is clear that these matrices form a group. Computation shows that all matrices of the form (2), (3) form a group also in the case when M and N are matrices of arbitrary order satisfying (3).

The methods by which Theorems 1-10 are proved turn out to be applicable also to the solution of certain problems in the theory of covariant tensors that are not polyvectors. We shall say that a covariant tensor K of valency k is obliquely divisible by a covariant vector α, and write $K \vdots \alpha$, if

$$Kx_1x_2 \dots x_k = 0$$

for any vectors x_i for which $\alpha x_i = 0$ $(i = 1, 2, \dots, k)$. The following proposition generalizes Lemma 1.

Lemma 3. *Let $k = i_1 + i_2$, and let the covariant tensor K of valency k be skew-symmetric in the first i_1 and in the second i_2 arguments. The propositions:*

a) the tensor K can be represented in the form

$$Kx_1x_2 \dots x_k = \alpha x_1\, Sx_2x_3 \dots x_{i_1}x_{i_1+1} \dots x_k + \alpha x_{i_1+1}\, Lx_1x_2 \dots x_i x_{i_1+2} \dots x_k,$$

where S and L are certain tensors, α is a covector;

b)

$$\alpha x\, \alpha u\, Kx_1x_2 \dots x_i x_{i_1+1} x_{i_1+2} \dots x_k = 0;$$

c) $K \vdots \alpha$,

are equivalent.

Lemma 3 is readily generalized to tensors that are skew-symmetric in the arguments of each of any number of groups and, in particular, to tensors that do not possess skew-symmetry in any pair of arguments.

Although the question of how, for a given tensor K, to determine whether it has an oblique divisor remains open, Lemma 3 nevertheless finds some applications. Thus, the question of the possibility of representing a trivector A in the form

$$A = [\alpha B] + [\beta C] \quad (A = [\alpha B] + [\beta\gamma\delta])$$

immediately leads to the equation

$$\alpha s\, \alpha t\, Axyz\, Auvw\, Apqr = 0 \qquad (\alpha s\, \alpha t\, Axyz\, Auvw = 0) \tag{4}$$

with the unknown covariant vector α. Equations (4) were obtained in (^{3}) by computations. Application of Lemma 3 to the investigation of the equation considered by Sh. D. Trushin (^{5}),

$$\sum_{i=1}^{n} \omega_i^m xx \dots x\alpha_i x = 0 \tag{5}$$

leads to the following result.

Theorem 11. The general solution of equation (5) has the form

$$\overset{m}{\omega}_i xx\ldots x = \sum_{j=1}^{n} \overset{m-1}{\sigma}{}_{ij} xx\ldots x\alpha_{jx},$$

where $\overset{m-1}{\sigma}{}_{ij}$ are arbitrary covariant tensors of valence $(m-1)$ such that $\overset{m-1}{\sigma}{}_{ij} = -\overset{m-1}{\sigma}{}_{ji}$.

All proofs are carried out without the use of the axiom of dimensionality and, consequently, all the assertions given above remain valid also for an infinite-dimensional space (4).

In conclusion, I take the opportunity to express my deep gratitude to my scientific adviser Prof. A. M. Lopshits, whose help and friendly attention I constantly feel, for valuable conversations and discussions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.