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Abstract

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ON THE HOLOMORPHIC CONTINUATION OF CONTRIBUTIONS OF FEYNMAN DIAGRAMS

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1. Consider connected Feynman diagrams with 4 external lines, at each vertex of which 3 lines enter and which have no proper energy loops. We shall consider only scalar particles of equal masses; let the square of the mass be equal to unity. The contribution of an arbitrary diagram of order n can be represented in the form [(1)]

$$F(s,t) = \int_{T_l} \frac{\delta\left(\sum_{i=1}^{l} \alpha_l - 1\right) d\alpha_1 \dots d\alpha_l}{d^2(\alpha)\{Q(\alpha,s,t)\}^{2n-l-2}}, \quad T_l \left(\alpha \mid 0 \le \alpha_i \le 1,\ \sum_{i=1}^{l} \alpha_i \le 1\right), \tag{1}$$

where l is the number of internal lines; s, t are the Mandelstam variables. The function $Q(\alpha,s,t) < 0$ in the region $s < 4,\ t < 4,\ u = 4 - s - t < 4$ can be represented in the form

$$Q(\alpha,s,t) = \frac{f(\alpha)s + g(\alpha)t - k(\alpha)}{d(\alpha)}, \tag{2}$$

where the functions $f(\alpha)$, $g(\alpha)$, $k(\alpha)$, and $d(\alpha)$ are homogeneous polynomials. Under our assumptions concerning the Feynman diagrams, the function $d^{-2}(\alpha)$ is summable in T_l.

2. Let us for the time being restrict ourselves to those diagrams for which $f(\alpha) \ge 0$ and $g(\alpha) \ge 0$ in T_l (planar diagrams). In this case $Q(\alpha,s,t) < 0$ in the region $B(s,t \mid s < 4,\ t < 4)$ for $\alpha \in T_l$, and the function $F(s,t)$ is holomorphic in the domain D_0, which contains: 1) all complex points with $\operatorname{Im} s$ and $\operatorname{Im} t$ of the same sign and arbitrary $\operatorname{Re} s$ and $\operatorname{Re} t$; 2) all complex points lying on the analytic planes

$$as + (1-a)t = c \tag{3}$$

for $0 \leq a \leq 1,\ c < 4$; 3) the real points of the region B. In the domain D_0 the function $F(s,t)$ is defined by formula (1).

If the function $F(s,t)$ can be holomorphically continued from the domain D_0 to a domain D, which is the direct product of two complex s- and t-planes with the cuts $\operatorname{Im} s = 0,\ \operatorname{Re} s \geq 4,\ \operatorname{Im} t = 0,\ \operatorname{Re} t \geq 4$ removed, then we shall say that the function $F(s,t)$ is holomorphic on the physical sheet.

The function $F(s,t)$ can be holomorphically continued from the domain D_0 to any point of the domain D, with the possible exception of points lying on the analytic surfaces [2]

$$g(s,t) = 0, \quad g_{i_1 \dots i_m}(s,t) = g_i(s,t) = 0, \tag{4}$$

where $g(s,t) = 0$, $g_i(s,t)$ are polynomials with real coefficients. The surface $g(s,t) = 0$ is the Landau surface of the original diagram, and the surfaces $g_{i_1 \dots i_m}(s,t) = 0$ are the Landau surfaces of all reduced diagrams that are obtained from the original diagram by contracting the i_1-th, ..., i_m-th lines to a point [3].

By methods of the theory of functions of several complex variables it has been possible to prove that the function $F(s,t)$ can be holomorphically continued from the domain D_0 to the domain D, if no real points of intersection

planes (3) with the surfaces (4) as the parameter $c \geqslant 4$ increases do not become complex[4]. If outside the domain B the Landau curves (4) have no singular points, then this condition is satisfied when outside the domain B there are no convex branches of the curves (4) with negative slope.

3. In the present note this result will be strengthened in two directions: 1) it will be shown that some Landau surfaces (4) cannot be surfaces of singularities of the function $F(s,t)$ in the domain D; 2) the conditions which the remaining Landau surfaces must satisfy in order that the function $F(s,t)$ admit a holomorphic continuation from the domain D_0 to the domain D will be weakened. For the proof, methods of the theory of functions of several complex variables and of homology theory are used[2,5–7].

Before proceeding to the statement of the results, let us introduce some definitions. If a reduced diagram is obtained from the original one by contracting the i_1-th, ..., i_m-th lines to a point and, at the same time, the functions

$$f(\alpha)\big|_{\alpha_{i_1}=0,\dots,\alpha_{i_m}=0}, \qquad g(\alpha)\big|_{\alpha_{i_1}=0,\dots,\alpha_{i_m}=0}, \qquad \alpha \in T_l,$$

are not identically zero simultaneously, then we shall call this reduced diagram an essential reduced diagram of the original diagram. If, however, the functions $f(\alpha)\big|_{\alpha_{i_1}=0,\dots,\alpha_{i_m}=0}$ and $g(\alpha)\big|_{\alpha_{i_1}=0,\dots,\alpha_{i_m}=0}$, $\alpha \in T_l$, are identically zero, then we shall call the reduced diagram an inessential reduced diagram of the original diagram.

The function $Q(\alpha, s, t)$ satisfies the equality

$$Q(\alpha, s, t)\big|_{\alpha_{i_1}=0,\dots,\alpha_{i_m}=0} = Q_{i_1\dots i_m}(\alpha, s, t) = Q_i(\alpha, s, t), \tag{5}$$

where $Q_i(\alpha, s, t)$ is the corresponding function of the reduced diagram. The function $Q_i(\alpha, s, t) < 0$ in the domain $B_i(s, t \mid s < m_i^2,\ t < n_i^2)$, where m_i and n_i are integers, $m_i \geqslant 2$, $n_i \geqslant 2$, determined by the reduced diagram.

We shall say that the Landau curve $g_i(s, t) = 0$ behaves normally if outside the domain B_i it has no singular points, and those of its branches which have negative slope and are located outside the domain B_i are concave. The following principal theorem holds.*

Theorem. *If all the Landau curves of the original diagram and of its essential reduced diagrams behave normally, then the function $F(s, t)$ can be holomorphically continued from the domain D_0 to the domain D.*

For lack of space we do not give the proof here.

We have given here not the most general formulation of the theorem, but in concrete examples of diagrams precisely this geometric picture of the Landau curves is realized.

4. The case in which the functions $f(\alpha)$ and $g(\alpha)$ are sign-changing reduces to the one considered by means of Lemma 1 of paper (4).

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CITED LITERATURE

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* It is assumed that the surfaces $f(\alpha)(1-a)-g(\alpha)a=0,\ 0<a<1$, have no singular points, and that the Landau equations are nondegenerate.

Note: Figure translations are in progress. See original paper for figures.