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Abstract

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MATHEMATICS

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ON THE LOCAL STRUCTURE OF CONTINUOUS MARKOV PROCESSES

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The first continuous Markov processes considered by A. N. Kolmogorov (^{1}) were diffusion Markov processes. Subsequently, W. Feller (^{2}) found the general form of continuous Markov processes on the line. Among these processes there turned out to be also some that were not diffusions. E. B. Dynkin, however, showed (see, for example, (^{3})) that they can be obtained from diffusions by rather simple transformations; in particular, nonterminating processes can be obtained from diffusions by changing the spatial variable and by a random change of time. In (^{4}) it was shown that multidimensional Markov processes that are martingales can be obtained by a random change of time from diffusion ones. In the present note we continue to study the connection between continuous processes and diffusion Markov processes. In what follows we use the terminology and notation of the book by E. B. Dynkin (^{5}).

1. Let X be a locally compact space, and let U be some neighborhood of X having compact closure. We shall consider a continuous strictly Markov process $\{x_t, \zeta, \mathfrak{M}_t, P_x\}$ in the space X. Since we shall be interested only in the local properties of the process, and a process with terminating trajectories can be obtained from a nonterminating process by shortening its lifetime, we shall assume that the killing time ζ coincides with the time of first exit from U. We shall also assume that the family of σ-algebras $\mathfrak{M}_t$ coincides with the intersection, over all measures P_x, of the completions of the σ-algebra generated by the quantities x_s for $s \leq t$. One of the principal tools in the investigation of the process will be the M-functionals of the process, defined below.

Definition. An additive almost homogeneous functional a_t is called an M-functional if it satisfies the following conditions:
1) $M_x a_t = 0$ for $x \in X$;
2) $M_x a_t^2$ is a measurable bounded function;
3) a_t is a continuous function of t with P_x-probability 1, whatever x may be;
4) there exists a positive additive almost homogeneous functional $\langle a, a\rangle_t$ for which $M_x a_t^2 = M_x \langle a, a\rangle_t$ for $x \in X$.

The most important example of an M-functional is the functional

$$\hat{\varphi}_t = \varphi(x_t) - \varphi(x_0) - \int_0^t A\varphi(x_s)\,ds, \qquad \varphi \in D_A,$$

where A is the infinitesimal operator of the process, and D_A is its domain of definition.

If α_t and β_t are M-functionals, then so is $\alpha_t + \beta_t$. With each pair of M-functionals α_t and β_t we associate the additive functional $\langle \alpha, \beta \rangle_t$:

$$\langle \alpha, \beta \rangle_t = {}^1/_2[\langle \alpha+\beta, \alpha+\beta \rangle_t - \langle \alpha, \alpha \rangle_t - \langle \beta, \beta \rangle_t].$$

Theorem 1. Let a sequence of partitions of the interval $[0, t]$: $0 = t_{n0} < \cdots < t_{nn} = t$ be such that $\max_k[t_{nk+1} - t_{nk}] \to 0$. Then

$$\sum_{k=0}^{n-1} (\alpha_{t_{nk+1}} - \alpha_{t_{nk}})(\beta_{t_{nk+1}} - \beta_{t_{nk}}) \to \langle \alpha, \beta \rangle_t$$

in probability $\mathbf{P}_x$, whatever $x \in X$.

Theorem 2. For any two M-functionals α_t and β_t, the function $\langle \alpha, \beta \rangle_t$ is absolutely continuous with respect to $\langle \alpha, \alpha \rangle_t$ with probability $\mathbf{P}_x$, equal to 1 for all $x \in X$, i.e., there exists a function $\psi(s)$ such that, with probability $\mathbf{P}_x$ equal to 1, the relation

$$\langle \alpha, \beta \rangle_t = \int_0^t \psi(s)\, d\langle \alpha, \alpha \rangle_s$$

holds.

Remark. The function $\psi(s)$ has the form $g(x_s)$, where $g(x)$ is some measurable function.

Definition. Let α_t and β_t be two M-functionals and

$$\langle \alpha, \beta \rangle_t = \int_0^t g(x_s)\, d\langle \alpha, \alpha \rangle_s.$$

Then we shall call $g(x)$ the **derivative** of the functional β_t with respect to the functional α_t, and denote

$$g(x) = \frac{\partial \beta}{\partial \alpha}(x).$$

The existence of the derivative (possibly nonunique) follows from Theorem 2.

2. Let us now consider stochastic integrals with respect to M-functionals. If $g(x)$ is such a function that

$$\sup_x \mathbf{M}_x \int_0^t g(x_s)^2 \, d\langle\alpha,\alpha\rangle_s < \infty,$$

then by the usual method one can construct the integral $\int_0^t g(x_s)\, d\alpha_s$. However, this integral will be the limit, with respect to the measure $\mathbf{P}_x$, of certain integral sums and therefore will depend on x. It turns out that there exists an M-functional γ_t such that

$$\gamma_t = \int_0^t g(x_s)\, d\alpha_s \tag{1}$$

with probability $\mathbf{P}_x$ equal to 1 for all x. In what follows, by a stochastic integral we shall always mean this M-functional. If the functional γ_t is defined by relation (1), then

$$\langle\gamma,\gamma\rangle_t = \int_0^t g(x_s)^2 \, d\langle\alpha,\alpha\rangle_s.$$

In the study of M-functionals the following generalization of Itô’ s formula for stochastic integrals is useful.

Theorem 3. Let $\alpha_t^{(1)},\dots,\alpha_t^{(k)}$ be M-functionals, and let $g(t,\xi_1,\dots,\xi_k)$ be a function of $k+1$ real variables for which there exist, are bounded, and are continuous the derivatives $g'_t(t,\xi_1,\dots,\xi_k)$, $g'_{\xi_i}(t,\xi_1,\dots,\xi_k)$,

$\xi_{\xi_i\xi_j}(t,\xi_1,\dots,\xi_k)$, $i,j=1,\dots,k$. Then

$$\begin{aligned} g(t,\alpha_t^{(1)},\dots,\alpha_t^{(k)}) &= g(0,0,\dots,0) + \int_0^t g'_s(s,\alpha_s^{(1)},\dots,\alpha_s^{(k)})\, ds \\ &+ \sum_{i=1}^k \int_0^t g'_{\xi_i}(s,\alpha_s^{(1)},\dots,\alpha_s^{(k)})\, d\alpha_s^{(i)} + \frac{1}{2}\sum_{i,j=1}^k \int_0^t g''_{\xi_i\xi_j}(s,\alpha_s^{(1)},\dots,\alpha_s^{(k)})\, d\langle\alpha_i,\alpha_j\rangle_s. \end{aligned} \tag{2}$$

Corollary. Let $\varphi_1(x),\dots,\varphi_k(x)$ be functions from D_A; let $g(\xi_1,\dots,\xi_k)$ be a twice continuously differentiable function. Then

$$g(\varphi_1(x_t),\dots,\varphi_k(x_t)) - g(\varphi_1(x_0),\dots,\varphi_k(x_0)) =$$

$$= \sum_{i=1}^{k} \int_0^t g'_{\xi_i}(\varphi_1(x_s), \dots, \varphi_k(x_s))\, d\hat{\varphi}_i(s) +$$

$$+ \sum_{i=1}^{k} \int_0^t g'_{\xi_i}(\varphi_1(x_s), \dots, \varphi_k(x_s))\, A\varphi_i(x_s)\, ds +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} \int_0^t g''_{\xi_i \xi_j}(\varphi_1(x_s), \dots, \varphi_k(x_s))\, d\langle \hat{\varphi}_i, \hat{\varphi}_j \rangle_s, \tag{3}$$

where

$$\hat{\varphi}_i(s) = \varphi_i(x_s) - \varphi_i(x_0) - \int_0^s A\varphi_i(x_u)\, du.$$

3. Introduce in the set of M-functionals the relation of subordination: the functional α_t is subordinate to β_t ($\alpha_t < \beta_t$), if

$$\alpha_t = \int_0^t g(x_s)\, d\beta_s,$$

where necessarily $g(x) = \dfrac{\partial \alpha}{\partial \beta}(x)$. The functional α_t is called maximal if the relation $\alpha_t < \gamma_t$ implies $\gamma_t < \alpha_t$.

Theorem 4. Every M-functional is subordinate to some maximal functional.

Theorem 5. If α_t and β_t are two maximal functionals, then the functions $\langle \alpha, \alpha \rangle_t$ and $\langle \beta, \beta \rangle_t$ are absolutely continuous with respect to one another.

Corollary. If α_t is an arbitrary M-functional, and $\bar{\alpha}_t$ is maximal, then there exists a function $g(x)$ for which

$$\langle \alpha, \alpha \rangle_t = \int_0^t g(x_s)\, d\langle \bar{\alpha}, \bar{\alpha} \rangle_s.$$

Take an arbitrary maximal functional $\bar{\alpha}_t$ and define τ_t from the relation $\langle \bar{\alpha}, \bar{\alpha} \rangle_{\tau_t} = t$. Put further $y_t = x_{\tau_t}$ (the process y_t is obtained from the process x_t by means of the random change of time generated by the functional $\langle \bar{\alpha}, \bar{\alpha} \rangle_t$). As follows from § 5, Ch. 10 of [5], y_t is also a continuous strong Markov process. It is easy to see that under such a change of time M-functionals pass into M-functionals, and maximal functionals into maximal functionals. Since for the maximal functional $\tilde{\alpha}_t = \bar{\alpha}_{\tau_t}$ we have $\langle \tilde{\alpha}, \tilde{\alpha} \rangle_t = t$, it follows that for any two M-functionals of the process y_t, β_t and γ_t, the expression $\langle\beta,\gamma\rangle_t$ will be absolutely continuous with respect to t, and hence

$$\langle\beta,\gamma\rangle_t = \int_0^t g(y_s)\,ds.$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be an arbitrary sequence of functions from the domain of definition of the infinitesimal operator $\tilde{A}$ of the process y_t, and let $g(\xi_1, \ldots, \xi_N)$ be a twice continuously differentiable function of its arguments. Denote by $\tilde{A}[\varphi_i\varphi_j](x)$ the function for which

$$\langle\hat{\varphi}_i, \hat{\varphi}_j\rangle_t = \int_0^t \tilde{A}[\varphi_i\varphi_j](y_s)\,ds.$$

Then from formula (3) there follows the relation

$$\mathbf{M}_y g(\varphi_1(y_t), \ldots, \varphi_N(y_t)) - g(\varphi_1(y), \ldots, \varphi_N(y_t)) =$$

$$= \int_0^t \left[\sum_{i=1}^N \frac{\partial g}{\partial\xi_i}(\varphi_1(y_s), \ldots, \varphi_N(y_s))\,A\varphi_i(y_s) + \right.$$

$$\left. + \frac{1}{2}\sum_{i,j=1}^k \frac{\partial^2 g}{\partial\xi_i\partial\xi_j}(\varphi_1(y_s), \ldots, \varphi_N(y_s))\,\tilde{A}[\varphi_i\varphi_j](y_s)\right] ds. \tag{4}$$

If we now choose the sequence φ_n so that it can be regarded as local coordinates in U, then it will follow from (4) that the quasicharacteristic operator of the process is a second-order differential operator, and this means precisely that the process y_t is quasidiffusive (see [5], p. 31).

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