A NUMERICAL METHOD FOR FINDING ASYMPTOTICALLY STABLE SOLUTIONS OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

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A. I. KONONENKO, V. D. MILMAN

A NUMERICAL METHOD FOR FINDING ASYMPTOTICALLY STABLE SOLUTIONS OF A SYSTEM OF ORDINARY DIFFEREN-TIAL EQUATIONS

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1. In the numerical solution of a system of equations

$$dx/dt = f(t, x) \qquad (f = \{f_i\}_{i=1}^n, \quad x = \{x_i\}_{i=1}^n) \tag{1}$$

a typical situation is one in which the desired solution is asymptotically stable, and the initial point is chosen correctly. At the same time, the fact that the presence of asymptotic stability eliminates accumulated errors is not taken into account, and the computation is carried out in such a way as to ensure, for large t, the required accuracy at each step. This shortcoming is partially overcome in the present note, in which algorithms are considered (of the Euler-method type) for finding a solution $x^0(t)$ that is asymptotically stable in some region* G $\left(\|x^0(t)\|^2 = \sum_{i=1}^n |x_i^0(t)|^2 < M\right)$ of system (1). We shall assume everywhere (and shall not mention this further) that the vector-function $f(t, x)$ has partial derivatives, uniformly bounded in t $(0 \le t < \infty)$, with respect to all arguments on every set H, where here and below H denotes a bounded set $H \subset G$.

Let us note that, in the indicated problem, in the case when $x^0(t) \equiv x^0$ (i.e., equation (1) has in G an asymptotically stable singular point), the question reduces to that of finding the minimum** of a continuously differentiable functional $F(x)$ having a unique stationary point in the region $G = \{x : F(x) < F(x_0)\}$, and also to the question of finding the saddle point*** of a convex-concave functional under special constraints. Thus, in the cases listed, Theorem 2 of the present note is applicable.

2. In proving convergence theorems, an essential role is played by a Lyapunov function $V(t, x)$. In what follows we shall use the fact that, under our assumptions concerning system (1), there exists a Lyapunov function $V(t, x)$ possessing the following properties****:

a) there exist continuous functions $W_i(x)$ $(i = 1, 2, 3)$ such that $W_3(x) \geq V(t, x) \geq W_1(x)$, where $W_3(x^0(t)) \equiv 0$, $W_1(x) > 0$ for $x \neq x^0(t)$

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* By the region of asymptotic stability is meant such a region G that, if $x_0 \in G$, then the trajectory $x(x_0, t_0, t) \in G$ for $t \geq t_0$.

** This case is discussed in the paper of I. M. Glazman ([1]). The methods of the present note are close to the methods of the indicated paper; in particular, the algorithms described in the note use so-called controlling sequences.

*** In ([2]) the indicated question is reduced to the problem of finding an asymptotically stable, in the whole space, singular point of a special differential equation.

**** A proof of this fact for the case when $x^0(t) \equiv x^0$ can be found, for example, in ([3]), pp. 37-38; the case of the trajectory $x^0(t)$ ($\|x^0(t)\| < M$) is easily reduced to the preceding case by the change of variables $y = x - x^0(t)$.

and $W_1(x) \to \infty$ as $x \to \overline{G} \setminus G$ (if G is unbounded, then $W_1(x) \to \infty$ when $\|x\| \to \infty$);

b)
$$dV(t, x)/dt = \partial V/\partial t + (\operatorname{grad}_x V, f(t, x)) < -W_2(x) < 0$$
for $x \neq x^0(t)$;

c) the function $V(t, x)$ has continuous partial derivatives of all orders with respect to all arguments, uniformly bounded in t on every $H \subset G$.

3. Let us first consider the case $x^0(t) \equiv x^0$. Let x_0 be an arbitrary point of the domain G, and let $\{\alpha_k\}_0^\infty$ be some sequence of nonnegative numbers such that
$$\sum_{n=0}^{\infty} \alpha_n = \infty.$$
Denote
$$t_n = \sum_{k=1}^{n-1} \alpha_k, \qquad t_0 = 0.$$
The point x_0 generates a sequence $\{x_n\}_0^\infty$ according to the rule
$$x_{n+1} = x_n + \alpha_n f(t_n, x_n). \tag{2}$$

Theorem 1. If the sequence $\{x_n\}_0^\infty$ defined above is contained in H ($x^0 \in H$) and $\alpha_n \to 0$ ($n \to \infty$), then $x_n \to x^0$ as $n \to \infty$.

The proof of the theorem is based on the following two lemmas, in which $V(t, x)$ is the Lyapunov function described above.

Lemma 1. Let $x^0 \in H$. Then there exists $\alpha > 0$ such that, for an arbitrary point $x \in H$ and $t \in [0, \infty)$,

$$V(t, x) > V(t + \alpha, x + \alpha f(t, x)). \tag{3}$$

Lemma 2. If $\alpha_n \to 0$ $(n \to \infty)$, the sequence $\{x_n\}_0^\infty$ defined by equations (2) with initial point $x_0 \in H$ is contained in H, and

$$\inf W_1(x_n) > 0,$$

then

$$\sum_{n=0}^{\infty} \alpha_n < \infty.$$

The use of Lemma 1 and of the properties of the Lyapunov function $V(t, x)$ shows that the following also holds.

Lemma 3. For every $x_0 \in G$ there exists $\alpha > 0$ such that the sequence $\{x_n\}_0^\infty$, $x_{n+1} = x_n + \alpha f(n\alpha, x_n)$, is contained in some set H.

4. We first describe the algorithms

$$\mathfrak{A}_1(\{A_n\}_1^\infty; \{\alpha_n\}_0^\infty) \quad \text{and} \quad \mathfrak{A}_1(\Phi_c; \{\alpha_n\}_0^\infty)$$

$$(0 < A_1 < ... < A_n < A_{n+1} < ... ; \ \alpha_n \geq 0; \ \Phi_c$$

denotes the set of x for which $\Phi(x) \leq c$); each of them consists of the successive performance of identically described stages (only the verification of the conditions for passing to the next stage differs). Stage number j consists in forming a sequence $x_n(j)$, $n = 0, 1, 2, ...$, according to the following rule:

$$x_{n+1}(j) = x_n(j) + \alpha_{n+j} f\left(\sum_{k=j}^{n+j-1} \alpha_k, x_n(j)\right),$$

where the initial point $x_0(j) = x_0$ (this point does not change when passing to the next stage). The construction of the sequence $\{x_n(j)\}_n$ is stopped, and the transition to the next stage is carried out, in the case of the algorithm

$$\mathfrak{A}_1(\{A_n\}_1^\infty; \{\alpha_n\}_0^\infty),$$

when the inequality $\|x_{n+1}(j)\| > A_j$ is satisfied, and, in the case of the algorithm

$$\mathfrak{A}_1(\Phi_c; \{\alpha_n\}_0^\infty),$$

when the inequality $\Phi(x_{n+1}(j)) > c$ is satisfied.

We shall also use the following algorithms:

$$\mathfrak{A}_2(\{A_n\}_1^\infty; \{S_n\}_0^\infty; \{\alpha_n\}_0^\infty) \quad \text{and} \quad \mathfrak{A}_2(\Phi_c; \{S_n\}_0^\infty; \{\alpha_n\}_0^\infty)$$

(here

$$0 = S_0 \le S_1 \le S_n \le S_{n+1} \le ...).$$

These algorithms also consist of identically described stages, and the conditions for transition to the next stage are the same as in the algorithms $\mathfrak{A}_1$. We describe the j-th stage of the algorithms $\mathfrak{A}_2$. Suppose that, from the initial point $x_0(j) = x_0$ (independent of the stage number), the points $\{x_k(j)\}_{k=1}^n$ have already been constructed and the first $i-1$ numbers $\{S_r\}_{r=1}^{i-1}$ have been crossed out*; then, if

$$S_i \le \sum_{k=1}^n \|x_k(j)-$$

* The rule for crossing out the numbers S_r is given below.

$-x_{k-1}(j)\|$, then the next point $x_{n+1}(j)$ is constructed by the formula

$$x_{n+1}(j) = x_n(j) + a_{i+j} f(t_n(j), x_n(j)); \tag{4}$$

where $t_n(j)$ is the sum of all α' s over the preceding steps of the j-th stage, and S_i is deleted from the sequence $\{S_n\}_0^\infty$. If, however,

$$S_i > \sum_{k=1}^n \|x_k(j) - x_{k-1}(j)\|,$$

then in (4) we write a_{i-1+j} instead of a_{i+j}, and S_i is not deleted. Upon passage to the next stage the entire sequence $\{S_i\}_0^\infty$ is restored.

Let us note that the algorithms $\mathfrak{A}_1$ are special cases of the algorithms $\mathfrak{A}_2$ (it suffices to take $S_n = 0$ for all n); nevertheless we have singled out their description, since they seem to us ideologically simpler.

5. From Theorem 1 and Lemma 3 it is not difficult to derive

Theorem 2. a) Let x^0 be an asymptotically stable, in the whole space, singular point of system (1). Then for any sequences $\{A_n\}_1^\infty$ ($A_n \to \infty$ as $n \to \infty$), $\{S_n\}_0^\infty$, and $\{a_n\}_0^\infty$ ($a_n \to 0$ as $n \to \infty$,

$$\sum_{=0}^\infty a_n = \infty)$$

and any initial point x_0, the algorithm $\mathfrak{A}_2(\{A_n\}_1^\infty; \{S_n\}_0^\infty; \{a_n\}_0^\infty)$ stabilizes at some stage j_0 and generates a sequence $\{x_n(j_0)\}_0^\infty$ converging to x^0.

b) Let x^0 be an asymptotically stable singular point in the t-uniformly bounded domain

$$H_t = \{x : \Phi(t,x) \le c\},$$

and let for every point x such that $\Phi(t,x) = c$ one have $(f(t,x), \mathrm{grad}_x \Phi(t,x)) < 0$. Then for any initial point $x_0 \in H_0$ and sequences $\{S_n\}_0^\infty$, $\{a_n\}_0^\infty$ ($a_n \to 0$, $\sum_{n=0}^\infty a_n = \infty$), the algorithm $\mathfrak{A}_2(\Phi_c; \{S_n\}_0^\infty; \{a_n\}_0^\infty)$ stabilizes at some stage j_0 and generates a sequence $\{x_n(j_0)\}_0^\infty$ converging to x^0.

In particular, Theorem 2 also holds in the case when $\{S_n = 0\}_{n=0}^\infty$, i.e., for the algorithms $\mathfrak{A}_1$.

6. Suppose that all solutions $x(x_0, t_0, t)$ of system (1) in some neighborhood of the point x^0 satisfy, for some $\beta > 0$, $B > 0$, the condition

$$\|x(x_0, t_0, t) - x^0\| \le B\|x_0\| \exp(-\beta(t - t_0)), \tag{5}$$

for $t \ge t_0$. Let us note that condition (5) is satisfied, for example, in the case when the right-hand sides of system (1) do not depend on t and, for the matrix of the linear part of the system at the singular point, all eigenvalues have negative real part (the system is nondegenerate).

Theorem 3. If system (1) satisfies condition (5), then under the conditions of Theorem 2 (both in case a) and in case b)), with the additional requirement $S_n \to \infty$, in the implementation of the algorithm $\mathfrak{A}_2$ only a finite set of numbers S_n is deleted, and thus the step α_n stabilizes.

In proving Theorem 3 we use the fact that, when condition (5) is fulfilled, the function $V(t,x)$ can be chosen to satisfy inequalities characteristic of a quadratic form (for details see (3), p. 75). Denote such a Lyapunov function by $V^0(t,x)$.

Lemma 4. For an arbitrary $H \subset G$ there exists a constant multiplier of lower relaxation, i.e., a number $\alpha_0 > 0$ such that for any α, $0 \le \alpha < \alpha_0$,

$$V^0(t + \alpha, x + \alpha f(t,x)) > V^0(t + \alpha_0, x + \alpha_0 f(t,x)).$$

In conclusion to this item, let us note that the intuitive idea that the rate of convergence of the process indicated in Theorem 2 decreases as the rate of convergence of the sequence $\{a_n\}_0^\infty$ to zero increases does not correspond to reality. Moreover, the following assertion holds.

For a monotone sequence $\{\alpha_n\}_0^\infty$, $\alpha_n \to 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$, one can find a system (1) and a sequence $\{\tilde{\alpha}_n\}_0^\infty$, $\sum_{n=0}^\infty \tilde{\alpha}_n = \infty$ and $\tilde{\alpha}_n/\alpha_n \to 0$, such that $\rho(\tilde{x}_n(j_0), x^0)/\rho(\tilde{\tilde{x}}_n(j_0), x^0) \to 0$ as $n \to \infty$, where the sequence $\{\tilde{x}_n(j_0)\}_{n=0}^\infty$ is constructed by the algorithm $\mathfrak{A}_1(\{A_n\}_1^\infty; \{\alpha_n\}_0^\infty)$ from an arbitrary point

$x_0 \neq x^0$, while $\{\tilde{\tilde{x}}_n(j_0)\}_0^\infty$ is constructed from the same initial point by the algorithm $\mathfrak{A}_1(\{A_n\}_1^\infty; \{\tilde{\alpha}_n\}_0^\infty)$, in which $\tilde{\alpha}_n > \alpha_n$ $(n = 0, 1, 2, ...)$.

7. In the general case of a bounded asymptotically stable trajectory $x^0(t)$, there is a theorem analogous to Theorem 2. Denote

$$\Omega(x(t),\, t \to \infty) := \{x :\ \exists\ (t_n \to \infty),\ x(t_n) \to x \text{ as } n \to \infty\};$$

we shall understand the notation $\Omega(x_n, n \to \infty)$ analogously.

Theorem 4. *The process indicated in Theorem 2 (both in the case of item a) and in the case of item b)) generates a sequence* $\{x_n(j_0)\}_0^\infty$ *converging to* $x^0(t)$ *in the sense that*

$$\Omega(x_n(j_0),\, n \to \infty) = \Omega(x^0(t),\, t \to \infty).$$

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Physical-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.