ON THE THEORY OF HOTELLING' S TEST

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Abstract

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ON THE THEORY OF HOTELLING' S TEST

Let $X_1, \dots, X_N$ be independent normal vectors of a p-dimensional Euclidean space, having common mean $\xi = EX_i$ (a p-dimensional column vector) and common correlation matrix

$$\Sigma = E'[(X_i - \xi)(X_i - \xi)^\tau],$$

$i = 1, \dots, N$; here τ denotes transposition. Put

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i;$$

$$S = \sum_{i=1}^{N}(X_i - \overline{X})(X_i - \overline{X})^\tau.$$

Consider the problem of testing the statistical hypothesis $H_0 : \xi = 0$ against the (composite) alternative $H_\delta : N\xi^\tau\Sigma^{-1} = \delta$, where δ is an arbitrary fixed positive number. Obviously, this problem may be interpreted, in a certain sense, as testing for the presence, in a given Gaussian noise with zero mean, of a signal of given energy. For its solution one usually uses Hotelling' s T^2-test (see ([1])), rejecting the hypothesis H_0 when

$$T^2 = N(N-1)\overline{X}^\tau S^{-1}\overline{X} > T_0^2,$$

where T_0^2 is a constant determined by the chosen level α. We note that for a constant value of δ the power of the T^2-test is also constant (see ([1])).

The statistical properties of the T^2-test were studied by J. Simaika ([2]), C. Stein ([3],[4]), N. Giri, J. Kiefer, and C. Stein ([5],[6]). The test has a simple structure, but its properties are mysterious. Up to now not a single nontrivial case is known for which the admissibility of the T^2-test has been proved. A weaker property here, minimaxity, was justified in ([6]) only for $p = 2$, $N = 3$. In this very interesting work the authors, relying on the Hunt-Stein theorem (see ([7])), reduce the problem of minimaxity of the T^2-test to the investigation of a Fredholm integral equation of the first kind, which for $p = 2$, $N = 3$ is transformed into an "overdetermined" linear differential equation of the first order; the explicit form of the solution of the latter makes it possible to prove the minimaxity of the T^2-test in this special case. In the general case the Fredholm integral equation has the form

$$\int_{\Gamma_1} \exp\left\{\sum_{j=1}^{p} t_j \sum_{i>j} \frac{\beta_i}{2}\right\} \cdot \prod_{i=1}^{p} \Phi\left(\frac{N-i+1}{2}, \frac{1}{2}, \frac{t_i\beta_i}{2}\right) d\lambda^*(\beta_1, \dots, \beta_p) = \Phi\left(\frac{N}{2}, \frac{p}{2}, \frac{\gamma}{2}\right),$$

where Γ_1 denotes the $(p-1)$-dimensional simplex

$$\{(\beta_1, \dots, \beta_p) : \ \beta_i \geq 0, \ \sum_{i=1}^{p} \beta_i = 1\}$$

and Φ is the confluent hypergeometric function. The equation must hold identically in $t_1, \dots, t_p$, satisfying the condition

$$\sum_{i=1}^{p} t_i = \gamma,$$

where γ is any fixed positive number, and the unknown function λ^* must be a probability measure on Γ_1. The minimaxity of the T^2-test is equivalent to the existence of such a λ^*.

In the present note we continue the investigation of N. Giri, J. Kiefer, and C. Stein and prove the minimaxity of the T^2-test under the alternative

H_δ for $p = 2$, $N = 4$. A slight complication of our arguments also gives the construction of an overdetermined boundary-value problem with a linear differential operator in the general case ($p = 2$).

For $p = 2$, $N = 4$ the problem consists in showing that on the interval $0 < \beta < 1$ there exists a probability density $\lambda(\beta)$ for which

$$\int_0^1 e^{t(1-\beta)/2} \Phi\left(2, \frac{1}{2}, \frac{t\beta}{2}\right) \Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{(\gamma-t)(1-\beta)}{2}\right) \lambda(\beta)\, d\beta = \Phi\left(2, 1, \frac{\gamma}{2}\right). \quad (1)$$

With a view to solving this problem, let us consider the equation

$$\int_0^1 e^{t(1-\beta)/2} \Phi\left(2, \frac{1}{2}, \frac{t\beta}{2}\right) \Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{(\gamma-t)(1-\beta)}{2}\right) \lambda(\beta)\, d\beta =$$

$$= \int_0^1 \Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{\gamma(1-\beta)}{2}\right) \lambda(\beta)\, d\beta. \quad (2)$$

Using the fact that

$$\Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{x}{2}\right) = (1+x)e^{x/2},$$

we rewrite it in the form

$$\int_0^1 e^{-\gamma\beta/2}\Phi\left(2,\frac{1}{2},\frac{t\beta}{2}\right)[1+\gamma-\gamma\beta-t(1-\beta)]\lambda(\beta)\,d\beta=$$

$$=\int_0^1 e^{-\gamma\beta/2}(1+\gamma-\gamma\beta)\lambda(\beta)\,d\beta.$$

The left-hand side of the last equality can be expanded in a power series in t and the system obtained

$$\int_0^1 e^{-\gamma\beta/2}\left[(r-2r^2)\beta^{r-1}+(1+\gamma+\gamma r+2r^2)\beta^r-(\gamma+\gamma r)\beta^{r+1}\right]\lambda(\beta)\,d\beta=0,$$

$$r=1,2,... \tag{3}$$

Let

$$e^{\gamma\beta/2}f(\beta)=\sum_{r=1}^{\infty}a_{r-1}\beta^{r-1}$$

be an arbitrary function representable by a series whose radius of convergence is greater than one. Multiply the r-th equation (3) by a_{r-1} and sum over r from 1 to ∞. After some calculations we find

$$\int_0^1\left[(2\beta^3-2\beta^2)f''+(-5\beta+6\beta^2-\gamma\beta^2+\gamma\beta^3)f'+\left(-1+3\beta-\frac{\gamma\beta}{2}+\gamma\beta^2\right)f\right]\lambda(\beta)\,d\beta=\int_0^1 L(f)\lambda(\beta)\,d\beta=0 \tag{4}$$

Choose a small $\varepsilon>0$ and apply Green' s formula to the differential form $L(f)$:

$$\int_\varepsilon^{1-\varepsilon}L(f)\lambda(\beta)\,d\beta=\int_\varepsilon^{1-\varepsilon}L^*(\lambda)f(\beta)\,d\beta+\mathcal{L}[f,\lambda]\Big|_\varepsilon^{1-\varepsilon},$$

where the adjoint form is

$$L^*(\lambda)=2\beta^2(\beta-1)\lambda''+\beta(-3+6\beta+\gamma\beta-\gamma\beta^2)\lambda'+\left(3\beta+\frac{3\gamma\beta}{2}-2\gamma\beta^2\right)\lambda$$

and the bilinear form is

$$\mathcal{L}[f,\lambda] = 2\beta^2(1-\beta)(f\lambda' - f'\lambda) - [\beta + \gamma\beta^2(1-\beta)]f\lambda.$$

From Frobenius theory (8) it follows that the equation $L^*(\lambda) = 0$ has two fundamental solutions in the interval $(0,1)$,

$$\lambda_{01} = \sum_{k=0}^{\infty} a_k\beta^k, \qquad \lambda_{02} = \beta^{-1/2}\sum_{k=0}^{\infty} c_k\beta^k \tag{5}$$

and two fundamental solutions

$$\lambda_{11} = \sum_{k=0}^{\infty} g_k(1-\beta)^k, \qquad \lambda_{12} = (1-\beta)^{-1/2}\sum_{k=0}^{\infty} h_k(1-\beta)^k, \tag{6}$$

where $a_0, c_0, g_0, h_0 \neq 0$. Hence any solution of the equation $L^*(\lambda) = 0$ is integrable on $[0,1]$. We assert, moreover, that there exists a solution of this equation for which

$$\lim_{\varepsilon\to 0} \mathcal{L}[f,\lambda]\Big|_{\varepsilon}^{1-\varepsilon} = 0. \tag{7}$$

Indeed, considering linear combinations of the solutions (5), and then (6), one can easily find that (7) holds if and only if $\lambda = \lambda_{12}$.

The preceding arguments allow us to conclude that $\lambda = \lambda_{12}$ is a solution of equation (4) and hence of equation (2). We now prove that $\lambda = \lambda_{12}$ does not vanish in the interval $(0,1)$.

The equation $L^*(\lambda) = 0$, after the substitution $\beta = 1-x$, $v(x) = \dfrac{1}{h_0}\sqrt{x}\lambda(1-x)$, becomes the equation

$$2x(x-1)v'' + (-1 + 4x - \gamma x + \gamma x^2)v' + (3/2 + 3\gamma x/2)v = 0, \tag{8}$$

and the subsequent substitution $u = -v'/v$ gives us the Riccati equation

$$u' = u^2 - \frac{1 - 4x + \gamma x - \gamma x^2}{2x(1-x)}u - \frac{3 + 3\gamma x}{4x(1-x)}. \tag{9}$$

Recall that

$$v(x) = \sum_{k=0}^{\infty} \frac{h_k}{h_0}x^k, \quad |x| < 1.$$

Suppose that the function $v(x)$ vanishes somewhere in the interval $(0,1)$. Since $v(0)=1$, among the roots of the equation $v(x)=0$ there will be a smallest one. Denote it by x_0. Then the function u is analytic in the disk $|x|<x_0$. From (8), $v'(0)=3/2$, i.e. $u(0)=-3/2$. Therefore $u(x)$ cannot vanish in the interval $(0,x_0)$, since at the point where this occurs for the first time one would have to have $u'(x)\geq 0$, but equation (9) shows that at this point $u'(x)<0$. The negativity of the function $u(x)$ leads to unbounded decrease as one approaches x_0 from the left ($v'(x_0)\neq 0$, since $v\not\equiv 0$), which again contradicts (9). All this means that $\lambda=\lambda_{12}$ has no zeros in the interval $(0,1)$.

Let us return to equation (2), which, as we have seen, is satisfied by the function $\lambda=\lambda_{12}\neq 0$. Using the method of Laplace transformations, it is not difficult to obtain the relation

$$\int_0^1 x^{b-1}(1-x)^{c-b-1}e^{qx}\Phi(a,b,px)\Phi(a-b,c-b,q(1-x))\,dx$$

$$=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\Phi(a,c,p+q),\qquad 0<b<c. \tag{10}$$

First putting $t=\gamma\tau$ in (2), we multiply both sides of this equality by $\tau^{-1/2}(1-\tau)^{-1/2}$ and integrate with respect to τ from 0 to 1. We find the integral with respect to τ, taking in (10) $a=2$, $b=1/2$, $c=1$, $p=\gamma\beta/2$, $q=\gamma(1-\beta)/2$. As a result it turns out that

$$\int_0^1 \Phi\left(2,1,\frac{\gamma}{2}\right)\lambda(\beta)\,d\beta=\int_0^1 \Phi\left(\frac{3}{2},\frac{1}{2},\frac{\gamma(1-\beta)}{2}\right)\lambda(\beta)\,d\beta.$$

To obtain (1), it suffices to use the homogeneity of the written relation and to normalize the function $\lambda=\lambda_{12}$ accordingly.

In conclusion we make a small remark. It can be shown that from the minimaxity of the T^2-test under any alternative H_δ, $\delta>0$, there follows the minimaxity of the T^2-test under the more natural alternative $H'_\delta: N\xi'\Sigma^{-1}\xi\geq\delta$. The proof is by contradiction, if one takes into account that, as we have already mentioned, for fixed δ the power of the T^2-test is constant. This remark is also substantiated by I. V. Romanovskii.

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Note: Figure translations are in progress. See original paper for figures.