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Abstract

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SYMMETRY OF A FORCE FIELD AND UNIVERSAL INTEGRALS OF MOTION

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By universal integrals we mean those integrals which preserve their form under a given symmetry of the force field, independently of the particular form of the potential. In what follows we shall consider only stationary fields. As Jeans noted as early as 1919 (*1*), the only integral that exists independently of the symmetry of the field is the energy integral I_1. In the case of rotational symmetry of the field with respect to some axis, in addition to the integral I_1, there exists a second universal integral I_2—the angular momentum with respect to this axis. Finally, in the case of spherical symmetry of the field there are also two universal integrals I_1 and I_2, and only in this case does I_2 correspond to an axis perpendicular to the plane of the orbit. All other integrals of motion will be peculiar, i.e., each existing only for one particular potential (as, for example, the elements of a Keplerian orbit exist only for the Newtonian potential). However, in order to determine the orbit for given initial conditions it is necessary to know not two, but three first integrals, and this is the reason for considerable difficulties in studying the general properties of orbits in fields with a given symmetry.

At first sight, the case in which the potential has the additive form stands apart,

$$U(x_1, x_2, x_3) = U_1(x_1) + U_2(x_2) + U_3(x_3), \tag{1}$$

and then there are three independent universal energy integrals

$$\tfrac{1}{2}\dot{x}_i^2 - U_i(x_i) = C_i, \qquad i = 1, 2, 3. \tag{2}$$

However, we shall show that even in the absence of additivity of the potential there exist three universal integral expressions analogous to integrals of motion. Namely, under a symmetry of the field the energy integral I_1 decomposes into the sum of two (or three) independent integral expressions of the indicated type, which we shall call **energy functionals with respect to the corresponding coordinate**.

For this purpose we shall prove the following theorem:

Theorem 1. *If a given stationary force field possesses symmetry with respect to some axis, then there exists an independent functional of the energy of motion parallel to this axis.*

Let x_3 be the axis of symmetry of the potential $U(x_1, x_2, x_3)$. Consider a test point M describing in the field of the potential U some orbit L. Along with the point M, consider the "antipodal" test point M', which at each instant has coordinates $(-x_1, -x_2, x_3)$. Obviously, M' will describe an orbit L' symmetric to the orbit L. Taking the masses of the points M and M' to be the same, the motion of their common center of gravity, which, ob-

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* Not necessarily rotational. For simple axial symmetry $U(x_1, x_2, x_3) = U(-x_1, -x_2, x_3)$; for rotational symmetry $U(x_1, x_2, x_3) = U(\sqrt{x_1^2 + x_2^2}, x_3)$.

as is evident, moves along the x_3 axis, will be described by the equations

$$x_1 = x_2 = 0, \quad \ddot{x}_3 = \partial U/\partial x_3. \tag{3}$$

Integrating the last of these, we obtain the desired energy functional

$$-\frac{1}{2}\dot{x}_3^2 - \int_L \frac{\partial U}{\partial x_3}\, dx_3 = C_3. \tag{4}$$

Let us note that here there appears an incomplete curvilinear integral, taken along a definite orbit L, which is a function only of its upper limit x_3. Thus, in a somewhat generalized form, the additivity of the potential with respect to the coordinates is expressed. Let us also note that, in the absence of axial symmetry, instead of the constant C_3 we would have an arbitrary function of x_1 and x_2. Finally, subtracting (4) from the energy integral, we obtain the second energy functional

$$\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \int_L \frac{\partial U}{\partial x_1}\, dx_1 + \frac{\partial U}{\partial x_2}\, dx_2 = C_2. \tag{5}$$

Symmetry with respect to two axes entails symmetry with respect to the third. In the case of three axes of symmetry we obtain, instead of one energy integral, three functionals

$$\frac{1}{2}\dot{x}_i^2 - \int_L \frac{\partial U}{\partial x_i}\, dx_i = C_i \quad (i = 1, 2, 3). \tag{6}$$

The functionals (6) have the feature that the unknown functions expressing the trajectory of the point L enter under the integral sign. Moreover, they are not single-valued, since an incomplete curvilinear integral over a closed contour is not equal to zero. On the other hand, however, the functionals (6) possess a number of properties that make them akin to integrals of motion of the usual type. First, their specification, together with the specification of the initial conditions, makes it possible to compute the orbit step by step by Euler' s method; second, in the case of the additive form of the potential (1) they become integrals of the usual type. The ordinary energy integral I_1 is always equal to their sum. We propose to call them **energy functionals**.

Since a force field with an axis of rotational symmetry and a plane of equatorial symmetry has practical importance in astrodynamics and in the dynamics of stellar systems–galaxies–we shall dwell on it separately. Let us note that rotational symmetry about an axis combined with equatorial symmetry is a special case of three-axis symmetry, since any two equatorial diameters may be taken as axes of symmetry.

Theorem 2. *The third universal integral, besides the energy integral I_1 and the angular-momentum integral I_2, can only be an energy functional; in this triad I_1 and I_2 can be replaced by two other energy functionals, and the result will be a universal triad equivalent to the first.**

Suppose that, in addition to the energy and angular-momentum integrals, there is also a third universal integral I_3. To simplify the proof, let us eliminate from I_1, with the aid of the angular-momentum integral I_2, the angular coordinate of the cylindrical system ρ, θ, z. Then two integrals remain

$$\frac{1}{2}(\dot{\rho}^2 + \dot{z}^2) - \int_L \frac{\partial U_1}{\partial \rho}\, d\rho + \frac{\partial U_1}{\partial z}\, dz = \frac{1}{2}(\dot{\rho}_0^2 + \dot{z}_0^2);$$

$$\varphi(\rho, z, \dot{\rho}, \dot{z}) = \varphi(\rho_0, z_0, \dot{\rho}_0, \dot{z}_0), \tag{7}$$

* In general form and in greater detail, the corresponding theorem has been proved elsewhere ([2]).

where $U_1 = U - \frac{1}{2}C_2^2/\rho^2$; φ stands in place of I_3, and $\rho_0, z_0, \dot{\rho}_0, \dot{z}_0$ are the initial values.

Let us single out in the four-dimensional phase space a small initial volume $d\gamma_0 = \Delta\rho_0 \Delta z_0 \Delta\dot{\rho}_0 \Delta\dot{z}_0$, whose edges are parallel to the coordinate lines, while $\Delta\rho_0$ and Δz_0 are proportional, respectively, to ρ_0 and $\dot{z}_0$. After a small interval of time dt, $d\gamma_0$ will pass into the volume $d\gamma = \Delta\rho\Delta z\Delta\dot{\rho}\Delta\dot{z}$, and, by Liouville' s theorem, $d\gamma_0 = d\gamma$; hence

$$\Delta\rho\Delta z/\Delta\rho_0\Delta z_0 = \Delta\dot{\rho}_0\Delta\dot{z}_0/\Delta\dot{\rho}\Delta\dot{z}. \tag{8}$$

But, according to our choice of edges,

$$\Delta\rho\Delta z/\Delta\rho_0\Delta z_0 = \dot\rho\dot z/\dot\rho_0\dot z_0 = 1 + (\ddot\rho_0/\dot\rho_0 + \ddot z_0/\dot z_0)\,dt \\ = 1 + dt\,[D\ln(\dot\rho\dot z)/Dt]_0, \tag{9}$$

where D/Dt is the Stokes derivative.

On the other hand, regarding (7) as equations substituting the variables $\dot\rho, \dot z$ by ρ_0, z_0, we find the Jacobian

$$\Delta\dot\rho_0\Delta\dot z_0/\Delta\dot\rho\Delta\dot z = (\dot\rho_0, \dot z_0)/(\dot\rho, \dot z) = 1 + dt\,[D\ln A/Dt]_0, \tag{10}$$

where

$$A = \begin{vmatrix} \dot\rho & \dot z \\ \partial\varphi/\partial\dot\rho & \partial\varphi/\partial\dot z \end{vmatrix} = \dot\rho\frac{\partial\varphi}{\partial z} - \dot z\frac{\partial\varphi}{\partial\rho}. \tag{11}$$

Substituting (9) and (10) into (8) and discarding the now unnecessary subscript, we obtain the condition

$$\frac{D}{Dt}\left[\ln\left(\frac{A}{\dot\rho\dot z}\right)\right] = \frac{D}{Dt}\left(\frac{A}{\dot\rho\dot z}\right) = 0. \tag{12}$$

In other words, the quantity

$$I = \frac{A}{\dot\rho\dot z} = \frac{1}{\dot z}\frac{\partial\varphi}{\partial z} - \frac{1}{\dot\rho}\frac{\partial\varphi}{\partial\rho} \tag{13}$$

must be either an integral of motion or a constant. Equating (13) in turn to arbitrary functions: of I_1, of the new "fourth" universal integral I_4, and, finally, of $I_3 = \varphi$ itself, we regard (13) as a partial differential equation with respect to φ.

The first two hypotheses immediately fall away, since φ turns out to be a function of I, or else one must admit the existence of yet another, fourth universal integral, which then would have to exist in all the simplest cases—for example, for an additive form of the potential $U(\rho, z) = U_1(\rho) + U_2(z)$, which in fact is not the case. In the last case we obtain the equation

$$\frac{\partial\varphi/\partial z}{\dot z} - \frac{\partial\varphi/\partial\dot\rho}{\dot\rho} = \frac{1}{F'(\varphi)},$$

where, for convenience, $F'(\varphi)$ denotes the derivative of a suitably chosen arbitrary function. The general solution of this equation has the form

$$1) \quad F(\varphi) = \frac{1}{2}\rho^2 + \Psi(\dot{\rho}^2 + \dot{z}^2) \qquad \text{or} \qquad 2) \quad F(\varphi) = \frac{1}{2}\dot{z}^2 + \Psi_1(\dot{\rho}^2 + \dot{z}^2),$$

where Ψ and Ψ_1 are arbitrary functions. Substituting these expressions into Boltzmann' s equation in order to determine Ψ and Ψ_1, we obtain, respectively,

$$1) \quad \Psi = -1 \quad \text{and} \quad \partial U_1/\partial z = 0 \qquad \text{or} \qquad 2) \quad \Psi_1 = -1 \quad \text{and} \quad \partial U_1/\partial \rho = 0.$$

It is easy to see that the conditions imposed on the potential are equivalent to the requirement that its form be additive. But precisely the introduction of incomplete curvilinear-

linear integrals and creates this additivity. Thus, the only integrals which, together with the classical ones, are capable of creating the triad of universal integrals necessary for investigating the general properties of systems with an axis of rotational symmetry and a plane of equatorial symmetry are the energy functionals described above.

The application of the functionals to problems of astrodynamics and the dynamics of galaxies will be given elsewhere. Here we shall only note that the energy functional in the z-coordinate means that the energy of stellar motions in galaxies in the z-coordinate can be specified independently of motions parallel to the equator. This is sufficient to explain the well-known division of the stellar population of galaxies into subsystems, and also to explain the observed three-axiality of the velocity ellipsoid, which contradicts the data of ellipsoidal theory.

The question of the multivaluedness of the energy functionals is subject to further consideration; however, it is in principle unavoidable, since it reflects the rosette-like character of orbits in symmetric fields of the given type.

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CITED LITERATURE

1. J. H. Jeans, *Problems of Cosmogony*, Cambridge, 1919;

2. K. F. Ogorodnikov, *Uch. zap. LGU*, Ser. Math., No. 328, 132 (1965).

Note: Figure translations are in progress. See original paper for figures.