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Abstract

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Corresponding Member of the Academy of Sciences of the USSR V. V. STRUMINSKII

ON THE SOLUTION OF THE CHAIN OF EQUATIONS OF THE KINETIC THEORY OF GASES

The first fundamental results on the justification of the kinetic theory of gases and, in particular, of the Boltzmann equation were obtained on the basis of the study of solutions of the chain of kinetic equations in works ($^{1-3}$). In recent years a number of new works have appeared in this direction, and on the whole the problem has been studied rather fully. However, some important aspects of it have remained insufficiently investigated. In particular, the processes leading to the loss of arbitrary initial data and to the multiplicativization of the distribution functions have not been sufficiently studied, as was pointed out by Uhlenbeck (^{4}). In the author's work (^{5}), where questions related to this problem are also considered, it is shown that in the equations for the s-particle distribution functions there appears a special integral type of interaction between molecules, not characteristic of classical mechanics and playing the principal role in the dynamics of irreversible processes. To construct irreversible solutions it is necessary already in the zeroth approximation to take into account the complicated dependence of the distribution functions on the fast and slow processes occurring in the system. For this purpose the paper uses a modified small-parameter method (^{6}), which made it possible to reduce the chain of kinetic equations to a recurrent system of differential equations.

In the present work a new method for solving the chain of kinetic equations is considered, and it is shown that the distribution functions depend in different ways on fast, reversible and slow irreversible processes. There exists a time interval $0 \leq t \leq \tau_1$ on which the influence of the slow processes is insignificant. Beyond it the slow processes will lead to an irreversible change of the distribution functions, to the loss of initial data. The slow processes themselves are connected with the multiplicative form of the distribution functions. Therefore, for $t > \tau_1$ the distribution function in the zeroth approximation loses the initial data and assumes a multiplicative form. This zeroth approximation is used in the work to construct the first and subsequent approximations for the correlation functions and to derive the kinetic equation.

Let us write the chain of kinetic equations in the form:

$$\partial f_s/\partial t = H_s f_s + \varepsilon \iint W_{s+1} f_{s+1}\, dx_{s+1}, \tag{1}$$

where ε is an auxiliary small parameter.

To solve (1) we shall use the method set forth in (^{5}), and put

$$f_s = f_s^{(0)} + \varepsilon f_s^{(1)} + \varepsilon^2 f_s^{(2)} + \dots, \tag{2}$$

$$f_s(t) = f_s(t, x_1, \dots, x_s, \omega_1, \dots, \omega_s). \tag{3}$$

In this case (1) reduces to the system of equations

$$\left(\frac{\partial f_s^{(0)}}{\partial t}\right)_\omega = H_s f_s^{(0)},$$

$$\left(\frac{\partial f_s^{(1)}}{\partial t}\right)_\omega = H_s f_s^{(1)} - A_k \frac{\partial f_s^{(0)}}{\partial \omega_k} + \iint W_{s+1} f_{s+1}^{(0)}\, dx_{s+1}, \tag{4}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

under the condition that

$$\frac{\partial \omega_k}{\partial t} = \left\langle \frac{df_1}{dt} \right\rangle_\xi = \varepsilon \iint W_2(\xi, v_k, x_2)\, f_2(t, \xi, v_k, x_2)\, dx_2. \tag{5}$$

From this expression we shall have

$$\omega_k(t, r_k, v_k) = f_1(t, r_k + v_k t, v_k), \tag{6}$$

$$A_k(t, r_k, v_k) = \iint W_2(\xi, v_k, x_2)\, f_2^{(0)}(t, \xi, v_k, x_2)\, dx_2. \tag{7}$$

. . . .

Let us now estimate the character of the variation of the functions ω_k with t. Expanding the dimensionless functions ω_k in a series, we find

$$\bar{\omega}_k(t, r_k, v_k) = \bar{\omega}_k(0, r_k, v_k) + t\,(\partial\bar{\omega}_k/\partial t)_0 + \cdots.$$

From expression (6) it follows that

$$\bar{\omega}_k(0, r_k, v_k) = f_1(0, r_k, v_k).$$

From expression (5) we obtain

$$\left(\frac{\partial\bar{\omega}_k}{\partial t}\right)_0 = \left(\frac{df_1}{dt}\right)_{t=0} = \frac{1}{\tau_\lambda}\iint \bar{W}_2\bar{f}_2\,d\bar{x}_2, \tag{8}$$

where $\tau_\lambda = \lambda/c$ is the relaxation time. Therefore we shall have

$$\bar{\omega}_k(t, r_k, v_k) = \bar{f}_1(0, r_k, v_k) + \frac{r_0}{\lambda}\,T\iint \bar{W}_2\bar{f}_2\,d\bar{x}_2, \tag{9}$$

where $t = \tau_0 T$ has been put. As is evident, in the zeroth approximation the functions ω_k do not depend on time ($\omega_k = \text{const}$). Dependence on time appears in the subsequent approximations. There exists a time interval $0 \le t \le \tau_1$ on which the set of functions ω_k may be regarded as constant, and their dependence on time will be substantial only for $t > \tau_1$.

In the zeroth approximation we have the equation

$$(\partial f_s^{(0)}/\partial t)_\omega = H_s f_s^{(0)} \tag{10}$$

with initial conditions in the form of an arbitrary function:

$$f_s^{(0)}(0) = f_s^{(0)}(0, x_1, \dots, x_s, \omega_1, \dots, \omega_s). \tag{11}$$

The dependence of the functions ω_k on t is not essential here, since over a certain time interval they remain constant and enter the initial conditions as parameters. The formal solution of (10) with the initial data (11) can be written in the form

$$f_s^{(0)}(t) = e^{tH_s} f_s^{(0)}(0) = f_s^{(0)}(0; X_k(t); \omega_k(R_k^{(s)}, P_k^{(s)}, t)). \tag{12}$$

As is evident, the character of the change of the distribution function in time is now determined by two processes: a) a rapidly proceeding, reversible process of change of $X_k(t)$ with duration of order $\tau_0 = r_0/c^*$; b) a slowly proceeding irreversible process of change of the functions ω_k with duration $\tau_\lambda > \tau_1 > \tau_0$.

Let, in accordance with what has been set forth, the functions ω_k remain constant in the interval $0 \le t \le \tau_1$. In this time interval expression (12) will give the exact, fully reversible solution of equation (10). It is usually assumed that this rapidly proceeding process irreversibly brings the distribution function to a multiplicative form, which is identified with the hypothesis of molecular chaos. In the general case such a process cannot change the initial form of the distribution function. In particular special cases one may expect multiplicativization of the distribution function, but this will always occur in a completely reversible manner and therefore has no direct relation to the hypothesis of molecular chaos. In the interval $0 \leqslant t \leqslant \tau_1$ the zeroth approximation will satisfy the Liouville equation (10) and will change in time in a completely reversible manner. In this case, as was shown in [5], all subsequent approximations will coincide with one another and will also lead to a completely reversible solution for the chain of equations (4).

Outside the time interval $0 \leqslant t \leqslant \tau_1$, the functions ω_k will begin to change noticeably in time, and this will have a very significant effect on the character of the further development of the system. For $t > \tau_1$ the distribution functions will change in time, in the general case, according to irreversible laws. As a result, there will occur a loss of initial data, a loss of initial information. The initial probability distribution can never again be reproduced. It is not hard to see that, in the presence of only one slow process, the distribution function will have a multiplicative form. Differentiating expression (12), we find

$$\frac{\partial f_s^{(0)}}{\partial t} = \left(\frac{\partial f_s^{(0)}}{\partial t}\right)_\omega + \frac{\partial f_s^{(0)}}{\partial \omega_k}\frac{\partial \omega_k}{\partial t}.$$

Expression (5) can always be represented in the form

$$\partial\omega_k/\partial t = \operatorname{div}_{v_k} \omega_k D_k.$$

From these two relations we obtain:

$$\frac{\partial f_s^{(0)}}{\partial t} = \left(\frac{\partial f_s^{(0)}}{\partial t}\right)_\omega + \left(\frac{\partial f_s^{(0)}}{\partial \omega_k}\right) \operatorname{div}_{v_k} \omega_k D_k.$$

If, for some $t > \tau_1$, $\left(\partial f_s^{(0)}/\partial t\right)_\omega = 0$, then the further change of $f_s^{(0)}(t)$ will depend only on the slow process and will be determined by the equation

$$\frac{\partial f_s^{(0)}}{\partial t} = \frac{\partial f_s^{(0)}}{\partial \omega_k} \operatorname{div}_{v_k} \omega_k D_k. \tag{13}$$

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* Here and throughout it is assumed that "capture" phenomena between molecules are practically absent.

Equation (13) has an exact solution of multiplicative form. Therefore, during some time interval $0 \leqslant t \leqslant \tau_2$, the process of multiplication of an arbitrary initial distribution function proceeds irreversibly:

$$f_s^{(0)}(t) = f_s^{(0)}(0, X_k(t), \omega_k(t)) = \prod_{1 \leqslant k \leqslant s} f_1(t, Q_k^{(s)}, P_k^{(s)}), \tag{14}$$

where

$$R_k^{(s)} = e^{tH_s} r_k, \quad P_k^{(s)} = e^{tH_s} p_k, \quad Q_k^{(s)} = R_k^{(s)} + \frac{t}{m} P_k^{(s)}. \tag{15}$$

In the general case the magnitude of this interval will depend on the initial form of the distribution function. For certain initial distribution functions of special form it may be that $\tau_2 \gg \tau_\lambda$. These cases require special study. However, for a number of dynamical systems, in particular for spatially homogeneous systems, and possible initial data, $\tau_2 \gtrsim \tau_1$. In what follows we shall consider only such systems.

In the first approximation the distribution function satisfies the equation:

$$\partial f_s^{(1)} / \partial t = H_s f_s^{(1)} + \psi_s^{(1)}. \tag{16}$$

From (4) and (7) we obtain an expression for $\psi_s^{(1)}$:

$$\psi_s^{(1)} = \sum_{k=1}^{s} \iint \left\{ U_{k\,s+1} f_{s+1}^{(0)} - \widetilde{U}_{k\,s+1} f_2^{(0)}(t, x_k, x_{s+1}) \frac{\partial f_s^{(0)}}{\partial \omega_k} \right\} dx_{s+1}.$$

We note that the first and second expressions under the sign of this integral have different structures.

If the distribution function is made subject to the condition

$$\begin{aligned} &\partial f_s^{(0)}(t, x_1, \dots, x_s, \omega_1, \dots, \omega_s) / \partial \omega_k = \\ &\quad = f_{s-1}^{(0)}(t, x_1, \dots, x_s, \omega_1, \dots, \omega_s), \end{aligned} \tag{17}$$

then these expressions will have the same structure. From (17) there follows unambiguously the necessity of multiplicativity of the distribution function in the zero approximation.

Taking, in accordance with the foregoing, that for $t > \tau_2$ the process of multiplication is complete, let us write the expression for $\psi_s^{(1)}$ in its final form

$$\psi_s^{(1)} = \iint \left\{ W_{s+1} \prod_{0<k<s+1} f_1(Q_k, P_k, t) - \widetilde{W}_{s+1} \prod_{1\leq k\leq s+1} f_1(\widetilde{Q}_k, \widetilde{P}_k, t) \right\} dx_{s+1}, \tag{18}$$

where the sign $\sim$ indicates that the coordinate r_k must be replaced by $r_k + v_k t$.

For the solution of the inhomogeneous equation (16), put:

$$f_s^{(1)}(t) = e^{tH_s} \varphi_s(t, x_k, \omega_k).$$

Substituting this expression into (16), we find

$$\partial\varphi_s/\partial t = e^{-tH_s} \psi_s^{(1)}(t). \tag{19}$$

Integrating equation (19), we obtain an expression for the first-approximation distribution functions satisfying zero initial data:

$$f_s^{(1)}(t) = \int_{\tau_2}^{t} e^{(t-\tau)H_s} \psi_s^{(1)}(\tau)\, d\tau. \tag{20}$$

Knowing the distribution function in the zero and first approximations, one can, in a completely analogous way, find the distribution function in the second approximation, and so on.

Thus, the modified small-parameter method has made it possible to find explicitly the irreversible solution for the chain of kinetic equations. For $s \geq 2$ we have

$$f_s(t, x_k, \omega_k) = \prod_{1\leq k\leq s} f_s(Q_k^{(s)}, P_k^{(s)}, t) + \varepsilon \int_{\tau_2}^{t} e^{(t-\tau)H_s} \psi_s^{(1)}(\tau)\, d\tau + O(\varepsilon^2). \tag{21}$$

The one-particle distribution function will satisfy a certain kinetic equation

$$\begin{aligned} \frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial r_1} = \varepsilon \iint W_2 \prod_{1\leq k\leq 2} f_1(Q_k^{(s)}, P_k^{(s)}, t)\, dx_2 + \\ + \varepsilon^2 \iint \left(W_2 \int_{\tau_2}^{t} e^{(t-\tau)H_2} \psi_2^{(1)}\, d\tau \right) dx_2 + O(\varepsilon^3). \end{aligned} \tag{22}$$

It can be shown that in the first approximation the kinetic equation (22) coincides with the generalized kinetic equation of N. N. Bogolyubov (^{1}).

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Note: Figure translations are in progress. See original paper for figures.