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Abstract

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MATHEMATICS

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THE GENERAL FORM OF A LINEAR FUNCTIONAL ON A BANACH SPACE OF ANALYTIC FUNCTIONS AND THE A-INTEGRAL

(Presented by Academician A. N. Kolmogorov on 15 V 1965)

In the present note a one-to-one functional representation is found for linear functionals on the Banach space of functions analytic inside the disk and continuous on its closure. The representation obtained leads to an extension of the concept of the Lebesgue integral, naturally connected with the problem of integrating functions harmonically conjugate to L-integrable functions*.

1. We consider the space E of functions $f(z)$, analytic inside the disk $|z| < 1$ and continuous on the disk $|z| \leqslant 1$, with norm

$$\|f\| = \max_{|z|\leqslant 1} |f(z)|.$$

E is isomorphically embedded in the space C of all complex-valued continuous functions on the circle $|z| = 1$. The space C' of all linear functionals on C, according to the Riesz theorem (^{1}), can be represented in the form of the direct sum

$$C' = l \oplus \tau \oplus L_1; \tag{1}$$

l, τ, and L_1 are respectively the spaces of discrete, singular, and absolutely continuous complex measures on the circle; moreover, for any $f \in C$ and $F \in C'$ the value $F(f)$ is computed by the formula:

$$F(f) = \int_0^{2\pi} f(t)(d\mu_d + d\mu_s) + (L)\int_0^{2\pi} f(t)\varphi(t)\,dt. \tag{2}$$

It turns out that the space of linear functionals on E admits an analogous representation:

Theorem 1. *The space E' of all linear functionals on E can be represented in the form of the direct sum*

$$E' = l \oplus \tau \oplus \left(L_1^r + \widetilde{L}_1^r\right), \tag{1}$$

where l, τ are respectively the Banach spaces of discrete and singular complex measures on the circle; L_1^r is the set of all real-valued L-integrable functions on it; $\widetilde{L}_1^r$ is the set of all functions harmonically conjugate to them; $L_1^r + \widetilde{L}_1^r$ is the Banach space of all real-valued functions representable in the form

$$r(t) = \varphi_1(t) + \widetilde{\varphi}_2(t) \quad \left(\varphi_1(t) \in L_1^r,\ \widetilde{\varphi}_2(t) \in \widetilde{L}_1^r\right). \tag{3}$$

In this case, for any functional $F \in E'$, the value $F(f)$ is computed by the formula

$$F(f) = \int_0^{2\pi} f(t)(d\mu_d + d\mu_s) + (A)\int_0^{2\pi} f(t)r(t)\,dt, \tag{4}$$

where μ_d, μ_s, and $r(t)$ are uniquely determined by the functional F, and the second integral is taken in the sense of A-integration (for the definition of the A-integral see, for example, (²)).

Proof of the theorem. On the basis of the theorem of F. Riesz and M. Riesz (³) one can prove that the set of all linear functionals from C' that vanish on E coincides with the space H_1, which, as is known (see, for example, (⁴)), is identified with the space of all functions L-integrable on the circle and analytically continuable into the disk. Consequently, $E' = C'/H_1$, and hence, since $C' = l \oplus \tau \oplus L_1$, while $H_1 \subset L_1$, the decomposition

$$E' = l \oplus \tau \oplus L_1/H_1. \tag{5}$$

is readily verified.

Let us find a representation of its third component L_1/H_1. To this end we shall use the following generalization of Riesz' s equality, found by P. L. Ul' yanov (⁵):

$$(L)\int_0^{2\pi} \psi(t)\,\tilde{f}(t)\,dt = -(A)\int_0^{2\pi} \widetilde{\psi}(t)f(t)\,dt \tag{6}$$

($f(t)$ is a real function measurable on $[0, 2\pi]$, bounded there together with its harmonic conjugate; $\psi(t)$ is L-integrable and real on $[0, 2\pi]$).

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* The principal results of this article were reported in April 1964 at Moscow University at a research seminar on function theory.

Let F be an arbitrary functional on E belonging to the component L_1/H_1; choose an arbitrary representative $\varphi(t)$ from the adjacent class determining F, denote $\operatorname{Re}\varphi = \varphi_1(t)$, $\operatorname{Im}\varphi = \varphi_2(t)$ ($\varphi_1, \varphi_2 \in L_1^r$), and associate with the functional F the real function $r(t)$ according to the rule

$$r(t) = \varphi_1(t) + \widetilde{\varphi}_2(t). \tag{7}$$

On the basis of (6) it is easy to verify that the value of the functional $F \in L_1/H_1$ can be expressed in terms of $r(t)$ by the formula

$$F(f) = (A)\int_0^{2\pi} f(t)r(t)\,dt. \tag{8}$$

We shall show that the real function $r(t) = \varphi_1(t) + \widetilde{\varphi}_2(t)$ associated with the functional $F \in L_1/H_1$ does not depend on the choice of the representative $\varphi(t)$ in the adjacent class F.

Indeed, choose another representative of the same class, $\varphi' = \varphi + h$, and form for it the real function $r'(t)$, according to the rule (7): $r' = \varphi_1' + \widetilde{\varphi}_2'$ ($\varphi_1' = \operatorname{Re}\varphi'$, $\varphi_2' = \operatorname{Im}\varphi'$); then $r' = \varphi_1 + \widetilde{\varphi}_2 + h_1 + \tilde{h}_2$, and since $h = h_1 + h_2 i \in H_1$, we have $h_2 = \tilde{h}_1$ and $\tilde{h}_2 = (\tilde{h}_1)^{\sim} = -h_1$, i.e. $r' = \varphi_1 + \widetilde{\varphi}_2 = r$. Consequently, the constructed mapping $L_1/H_1 \to L_1^r + \widetilde{L}_1^r$ (denote it by θ) is unique.

Similarly it is verified that to each real function $r(t)$ of the form $r(t) = \varphi_1(t) + \widetilde{\varphi}_2(t)$ ($\varphi_1, \varphi_2 \in L_1^r$) there corresponds a unique adjacent class $F = H_1 + (\varphi_1 + i\varphi_2) \in L_1/H_1$ such that $\theta(F) = r(t)$, i.e. θ is a one-to-one mapping of the set L_1/H_1 onto the set $L_1^r + \widetilde{L}_1^r$. This mapping transfers to the set of real functions $L_1^r + \widetilde{L}_1^r$ the structure of the complex vector space L_1/H_1. At the same time it is easy to see (see (7)) that the operations of addition and multiplication by real numbers in the functional space $L_1^r + \widetilde{L}_1^r$ are carried out in the same way as in the space L_1, while multiplication by the number i is there harmonic conjugation, i.e. $i \cdot r = \tilde{r}$. Moreover, the mapping θ transfers to the vector space $L_1^r + \widetilde{L}_1^r$ also the topology of the space L_1/H_1, i.e. the topology of the Banach space conjugate to the Banach space E. Thus, relying on the decomposition (5), we arrive at the decomposition (I)

space E', and on the basis of relations (2) and (8) we also find that the value of any functional on E is computed by formula (4).

Remark 1. It is important to note that the third component of the decomposition (5), uniquely representable, according to Theorem 1, in the class of A-integrable functions, does not admit a unique representation in the class of L-integrable functions. This follows from the fact that, as is easy to verify, the existence of such a representation would mean that the subspace H_1 is complemented in L_1 (a subspace $D' \subset D$ is called complemented in D if there exists a subspace $D_2 \subset D$ such that $D_1 \oplus D_2 = D$); but, as Newman showed (^{6}), this is not the case.

2. For the space E' of linear functionals on the space E, one can give another functional representation, somewhat more natural for it than (I),

$$E' = l \oplus \tau \oplus \widetilde{H}_1^A, \tag{II}$$

where $\widetilde{H}_1^A$ is the Banach space of all functions $h(z)$ representable in the domain $|z| > 1$ by a Cauchy-type L-integral; moreover, for any $f \in E$ and $F \in E'$, the value $F(f)$ is computed by the formula

$$F(f) = \int_0^{2\pi} f(t)(d\mu_d + d\mu_s) + (A) \int_0^{2\pi} f(t)\hat{h}(e^{it})\,dt, \tag{II'}$$

where $\mu_d \in l$, $\mu_s \in \tau$, and $h(z) \in \widetilde{H}_1^A$ are uniquely determined by the functional F, while $\hat{h}(e^{it})$ is the boundary value of the function $h(z)$ on the contour $|z| = 1$.

3. The concept of the A-integral was first introduced in probabilistic form by A. N. Kolmogorov (^{7}). The entry of the A-integral into function theory is connected with the problem of integrating functions harmonically conjugate to L-integrable ones: as Titchmarsh proposed (^{8}) and P. L. Ul' yanov proved (^{5}), every such function that is not in general L-integrable is A-integrable. However, the class of all A-integrable functions is very broad; for example, M. L. Bondy (^{9}) showed that every discontinuous function whose product with any A-integrable function is A-integrable is a constant. This class is too broad, in particular in the sense of the problem that called it forth: it is not closed with respect to the operator of harmonic conjugation; that is, as M. L. Bondy proved (^{9}), the function

$$\widetilde{\varphi}(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} (A) \int_\varepsilon^\pi \frac{\varphi(t+\alpha) - \varphi(t-\alpha)}{2\,\mathrm{tg}\,\alpha/2}\,d\alpha$$

is not in general A-integrable.

In this connection the following problem arises. Extend the concept of the Lebesgue integral by introducing an integral A_0 in such a way that the following requirements are satisfied:

1) The integral A_0 must have the property of additivity.
2) The class of A_0-integrable functions must contain all functions conjugate to L-integrable ones.
3) The class of A_0-integrable functions must be closed under the action of harmonic conjugation (regarded as a special A_0-integral).

4) The integral A_0 must be a minimal extension of the L-integral in the sense that, if A' is any extension of the L-integral possessing properties 1), 2), and 3), then the class of A'-integrable functions contains the class of A_0-integrable functions.

This problem, in a somewhat different form, was posed by P. L. Ul' yanov ([10]).

We introduce the following definition: a function $r(t)$, measurable on the interval $[0, 2\pi] = I$, is called A_0-integrable on I if there exists a number $C > 0$ such that for every trigonometric polynomial $Q(t)$

the inequalities hold

$$\left|(L)\int_0^{2\pi}[r(t)]_{-N}^{N}Q(t)\,dt\right| \le C\max_{t\in I}\left[\,|Q(t)|,\ |\widetilde{Q}(t)|\,\right], \qquad N = 1, 2, \dots,$$

and the number

$$\lim_N \int_0^{2\pi}[r(t)]_{-N}^{N}\,dt$$

is called the A_0-integral of the function $r(t)$.

Theorem 2. The A_0-integral defined above is a solution of the problem posed; moreover, the class of A_0-integrable functions coincides with the space $L_1^r + \widetilde{L}_1^r$,

$$(A_0)\int_0^{2\pi} r(t)\,dt = \lim_N \int_0^{2\pi}[r(t)]_{-N}^{N}\,dt = (A)\int_0^{2\pi} r(t)\,dt$$

for every function $r(t) \in L_1^r + \widetilde{L}_1^r$.

The proof of Theorem 2 is based on Theorem 1.

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References

[1] F. Riesz, C. R., **149**, 974 (1909).
[2] P. L. Ulyanov, DAN, **102**, No. 6, 1077 (1955).
[3] F. Riesz, M. Riesz, IV Skand. Mat. Kongr., Stockholm, 1916, C. R., Uppsala, 1920.
[4] K. Hoffman, *Banach Spaces of Analytic Functions*, IL, 1963.
[5] P. L. Ulyanov, Uch. zap. MGU, **8**, issue 181, 139, 1956.
[6] D. J. Newman, Proc. Am. Math. Soc., **12**, 98 (1961).
[7] A. N. Kolmogorov, *Basic Concepts of Probability Theory*, M.-L., ch. VI, 1936.
[8] E. C. Titchmarsh, Proc. London Math. Soc., **29** (1929).

[9] I. L. Bondi, UMN, **18**, issue 1 (1963).
[10] P. L. Ulyanov, UMN, **19**, issue 1 (1964).

Note: Figure translations are in progress. See original paper for figures.