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Abstract

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HYDROMECHANICS

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ON THE INFLUENCE OF FLUCTUATIONS OF ENERGY DISSIPATION ON THE FORM OF TURBULENCE CHARACTERISTICS IN THE INERTIAL RANGE

(Presented by Academician A. N. Kolmogorov on 5 V 1965)

In the modern theory of the local structure of developed turbulence, the central place is occupied by the "two-thirds law" found by A. N. Kolmogorov [1,2] and A. M. Obukhov [3] for the longitudinal and transverse structure functions of the velocity field $D_{LL}(r)$ and $D_{NN}(r)$ in the inertial range $L \gg r \gg \eta$ (where L and η are the outer and inner scales of turbulence), and by the equivalent "five-thirds law" for the velocity spectrum $E(k)$ in the range $1/L \ll k \ll 1/\eta$. However, almost immediately after the appearance of papers [1–3], L. D. Landau noted that the indicated laws cannot be completely exact because of the presence of random fluctuations of the quantity

$$\varepsilon = \frac{\nu}{2} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 , \tag{1}$$

which determines the rate of dissipation of kinetic energy (see, for example, the footnote on p. 117 of the book [4]). Nevertheless, for a number of years no attention was paid to this remark. During that time the "two-thirds" and "five-thirds" laws were repeatedly tested experimentally, and it turned out that they are confirmed with great accuracy, so that deviations from them, if they do exist in reality, must be very small.

Recently, in the works of A. M. Obukhov [5] and A. N. Kolmogorov [6], a generalization was outlined of the basic ideas concerning the similarity of small-scale components of turbulence at large Reynolds numbers, connected with taking into account the fluctuations of ε, and the influence of these fluctuations on the form of the statistical characteristics of the velocity field was discussed. The appearance of these works again drew attention to the question of fluctuations of energy dissipation. In particular, A. S. Gurvich and S. L. Zubkovskii [7] measured in the near-ground layer of air, over a wide interval of frequencies, the temporal spectrum of the fluctuations of the quantity

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$$\varepsilon_1 = \left\langle \frac{1}{\bar{u}_1}\frac{\partial u_3}{\partial t}\right\rangle^2 \simeq \left\langle \frac{\partial u_3}{\partial x_1}\right\rangle^2$$

(where the axis Ox_3 is directed upward, and the axis Ox_1 is along the mean wind $\bar{u}_1$), which is related to the dissipation ε. It turned out that the results they obtained, after passing from the temporal spectrum to the one-dimensional spatial spectrum $E^{(1)}_{\varepsilon_1\varepsilon_1}(k)$ of the field $\varepsilon_1(\mathbf{x}, t)$ by using Taylor' s hypothesis of "frozen turbulence" (which is fulfilled with high accuracy in the frequency range under consideration), for $0.02\ \mathrm{m}^{-1} \le k \le 100\ \mathrm{m}^{-1}$ are well described by the relation:

$$E^{(1)}_{\varepsilon_1\varepsilon_1}(k) \sim k^{-0.6}. \tag{2}$$

To be sure, result (2) is rather crude, since in the measurements described in [(7)] the value of $\partial u_3/\partial x_1$ at a point was in fact replaced by a value averaged over a parallelepiped with dimensions along the axes Ox_1, Ox_2, and Ox_3 of about 5, 0.5, and 4 cm, respectively. However, recently practi-

essentially the same result was obtained by Pond and Stewart ([8]) for the very close quantity $\varepsilon_1 = (\partial u_1/\partial x_1)^2$, and by means of fine thermoanemometric measurements using much less averaging (not distorting the disturbances responsible for energy dissipation).

Result (2) can in no way be reconciled with the formula following from the calculation ([9]) based on M. D. Millionshchikov' s hypothesis that the fourth-order semi-invariants of the velocity field are equal to zero. In this connection, E. A. Novikov and R. W. Stewart ([10]) proposed a new model of the small-scale structure of the energy-dissipation field $\varepsilon(\mathbf{x}, t)$ (taking into account the important property, known from observations, of the "intermittency" of small-scale velocity pulsations), in which the spectrum $E^{(1)}_{\varepsilon\varepsilon}(k)$ was already proportional to $k^{-\alpha}$, where $0 < \alpha < 1$.

The purpose of the present note is to clarify that a formula of the form (2) can also be derived from the assumptions contained in ([5],[6]), and that, in doing so, it can be connected with expressions for the deviations of the statistical characteristics of turbulence in the inertial interval from the "two-thirds" and "five-thirds" laws, allowing (at least in principle) independent experimental verification.

In ([5],[6]) it was assumed that the quantity ε_r, obtained from $\varepsilon(\mathbf{x}, t)$ by spatial averaging over the volume of a fixed shape with linear dimension r, for large Re and $L/r \gg 1$, has a logarithmically normal probability distribution. In this case, for the variance of the quantity $\log \varepsilon_r$, the asymptotic formula indicated in ([6]) was

$$\delta^2_{\log \varepsilon_r} \approx A(\mathbf{x}, t) + \mu \log L/r, \tag{3}$$

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where the term $A(\mathbf{x}, t)$ was assumed to depend on the characteristics of large-scale motions, while μ was regarded as having a universal value. Formula (3) and the assumption of a logarithmically normal distribution of ε_r correspond to a definite statistical model of dissipation fluctuations, which may be described as follows.

Let us take into account that the outer scale of turbulence L coincides with the characteristic distance over which the mean hydrodynamic fields, and in particular the field of mean energy dissipation $\varepsilon(\mathbf{x}, t)$, change appreciably. Consider the turbulent motion within a cube V_0 with edge L. The dissipation ε_0, averaged over the volume of this cube, may be identified with $\varepsilon(\mathbf{x}, t)$; it will be determined by the characteristics of the large-scale motion and will specify the mean amount of energy transferred in the cube V_0 from the mean motion to turbulent pulsations per unit time per unit mass of fluid.

Following the method used in ([10]), let us next divide the cube V_0 into an arbitrary number n of "first-order cubes" V_1 with edges $L_1 = L_0 n^{-1/3}$; then divide each of the "first-order cubes" into n "second-order cubes" V_2 (with edges $L_2 = L_1 n^{-1/3} = L n^{-2/3}$), and so on. In this case the energy dissipation averaged over the volume of one of the cubes of order j will be a random quantity ε_j with mean value $\bar{\varepsilon}_j = \varepsilon_0 = \varepsilon$. It is clear that the process of successive subdivision of cubes may be regarded, to some extent, as corresponding to the "cascade process" of generation of ever smaller turbulent formations, whose consideration lies at the basis of the entire theory of local structure. On the basis of the self-similarity of this cascade process, it seems natural to assume that, for $L_j = L n^{-j/3} \ll L$, the conditional probability distribution of the energy dissipation ε_j, averaged over the volume of a cube of order j, under the condition that the value of the dissipation ε_{j-1}, averaged over the volume of the cube of order $(j-1)$ containing it, is fixed, will be the same for all cubes of order j and will not depend on the number j until molecular viscosity begins to have a direct effect. It follows that if we denote by the symbol $e_j = \varepsilon_j(\varepsilon_{j-1})$ the random quantity

$\varepsilon_j/\varepsilon_{j-1}$ for a fixed value of ε_{j-1}, then

$$\varepsilon_j = \bar{\varepsilon} e_1 e_2 \dots e_j, \qquad \log \varepsilon_j = \log \bar{\varepsilon} + \sum_{k=1}^{j} \log e_k, \tag{4}$$

where all the terms $\log e_k$, except for several of the first ones, are identically distributed independent random variables. We shall further assume that the quantities $\log e_j$ have a finite mean value and a finite variance (this assumption distinguishes the model under consideration from that used in [10], where it was assumed that the event $e_k = 0$ has a finite probability close to unity). In that case, by the central limit theorem of probability theory, $\log \varepsilon_j$ for $j \gg 1$ must have a normal probability distribution. The mean value m_j and the variance σ_j^2 of the quantity $\log \varepsilon_j$ for $j \gg 1$ (but $L_j = L n^{-j/3}$ still appreciably exceeding

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the distances at which molecular viscosity begins to be felt) can, by virtue of (4), be represented in the form

$$m_j = \log \bar{\varepsilon} + A_1(\mathbf{x}, t) + jm, \qquad \sigma_j^2 = A(\mathbf{x}, t) + j\sigma^2. \tag{5}$$

Here m and σ^2 are the mean value and variance of the quantities $\log e_k$ with sufficiently large number k, while the terms $A(\mathbf{x}, t)$ and $A_1(\mathbf{x}, t)$ (arising because of the nonuniversality of several of the first terms $\log e_k$) depend on the characteristics of the large-scale motions (and change appreciably only when the point $\mathbf{x}$ is displaced by a distance of order L). But ε_j, obviously, may also be denoted as ε_r, where $r = L_j = Ln^{-j/3}$, i.e. $j = 3\log(L/r)/\log n$. Hence relation (3) is obtained, with $\mu = 3\sigma^2/\log n > 0$.

The argument given, of course, does not depend on the choice of the special cubic form of the volumes V_0, V_1, etc., and may be applied to volumes of any shape. The principal consequences of (3), presented in paper [6], are as follows. First of all, from the general formula for the moments of a log-normal distribution and from the fact that $\bar{\varepsilon}_r = \varepsilon$ for all $r \ll L$, we obtain

$$\overline{\varepsilon_r^p} = B_p(\mathbf{x}, t)\bar{\varepsilon}^p (L/r)^{p(p-1)\mu/2}, \tag{6}$$

where $\log B_p(\mathbf{x}, t) = p(p-1)A(\mathbf{x}, t)/2$. Hence, in particular, it is seen that for $r \ll L$ the variability of the quantity ε_r is very great, so that the intensity of the pulsating motion in different volumes of one and the same size r will be sharply different. This circumstance should outwardly manifest itself as "intermittency" of small-scale turbulence. Let us now assume, following [5,6], that the conditional probability distribution for the velocity differences at a distance r, appreciably exceeding the greatest distance at which viscosity is still directly felt, provided that the value of ε_r is fixed, depends only on r and ε_r. Then, by dimensional considerations, the conditional mean values (for fixed ε_r) of the squares of the differences of the longitudinal and transverse components of the velocity at a distance r must be proportional to $(\varepsilon_r r)^{2/3}$ (cf. (1)). If it is assumed that $r \ll L$, but that for the random variable $\mathrm{Re}_r = \varepsilon_r^{1/3} r^{4/3}/\nu$ (playing the role of the Reynolds number for turbulent motions of scale r) the relation $\mathrm{Re}_r \gg 1$ may fail to hold only with negligibly small probability, then $\overline{\varepsilon_r^{2/3}}$ will be determined by formula (6). Hence, for the unconditional mean values of the squares of the velocity differences we obtain the formulas

$$D_{LL}(r) = C\bar{\varepsilon}^{2/3} r^{2/3} (L/r)^{-\mu/9}, \qquad D_{NN}(r) = C_1 \bar{\varepsilon}^{2/3} r^{2/3} (L/r)^{-\mu/9}, \tag{7}$$

where the coefficients $C = C(\mathbf{x}, t)$ and $C_1 = C_1(\mathbf{x}, t)$ may depend on the macrostructure of the flow. With the aid of formulas relating the structure functions to the spectrum $E(k)$, this also yields the result

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$$E(k) = C_2 \bar{\varepsilon}^{2/3} k^{-5/3} (Lk)^{-\mu/9}. \tag{8}$$

for $1/L \ll k \ll 1/\eta_0$, where η_0 is the upper bound of the values actually attained by the random scale $\eta = \nu^{3/4}\varepsilon^{-1/4}$.

Let us now calculate the correlation function $B_{\varepsilon\varepsilon}(r) = \overline{\varepsilon(\mathbf{x}+\mathbf{r})\varepsilon(\mathbf{x})}$ (where $r = |\mathbf{r}|$) of the field $\varepsilon(\mathbf{x}, t)$ corresponding to the scheme under consideration of successive independent fragmentations of turbulent formations. For definiteness we shall again start from the simplest scheme of fragmentation of a cube of side L into ever smaller cubes with sides $L_j = Ln^{-j/3}$. In this case the value of the dissipation at a point will be determined by the equality $\varepsilon = \bar{\varepsilon} e_1 e_2 \dots e_s$, where the last factor e_s already corresponds to so small a cube that, within it, fluctuations of the dissipation may be neglected. But the points $\mathbf{x}+\mathbf{r}$ and $\mathbf{x}$ may be regarded as lying in one cube of order

$$j = 3 \log \frac{L}{r} \Big/ \log n,$$

but in different cubes of all subsequent orders. In this case the conditional mean value of the product $\varepsilon(\mathbf{x}+\mathbf{r})\varepsilon(\mathbf{x})$, for a fixed value $\varepsilon_r = \varepsilon_j$, will differ from ε_j^2 only by the factor $\overline{e'_{j+1} e_{j+1}}$, where e'_{j+1} and e_{j+1} are the values of the ratio $\varepsilon_{j+1}/\varepsilon_j$ for the cubes of order $(j+1)$ containing the points $\mathbf{x}+\mathbf{r}$ and $\mathbf{x}$. Consequently, $\overline{\varepsilon(\mathbf{x}+\mathbf{r})\varepsilon(\mathbf{x})}/\varepsilon_r^2 = \text{const}$ for $L \gg r \gg \eta_0$, i.e.

$$B_{\varepsilon\varepsilon}(r) = \overline{\varepsilon(\mathbf{x}+\mathbf{r})\varepsilon(\mathbf{x})} = \text{const}\, \overline{\varepsilon_r^2} = B\bar{\varepsilon}^2 (L/r)^{\mu}. \tag{9}$$

(for the one-dimensional model this result also follows at once from the formula

$$B_{\varepsilon\varepsilon}(r) = \frac{1}{2} \frac{d^2}{dr^2} (r^2 \overline{\varepsilon_r^2}),$$

which relates the correlation function of a stationary process $\varepsilon(x)$ to the mean square $\overline{\varepsilon_r^2}$ of the value of this process averaged over an interval of length r). To the correlation function (9) in the inertial interval of wave numbers there corresponds a one-dimensional spectrum of the form

$$E^{(1)}_{\varepsilon\varepsilon}(k) = B' \bar{\varepsilon}^2 L (kL)^{-1+\mu} \sim k^{-1+\mu}. \tag{10}$$

The preceding arguments may be applied not only to the quantity ε in formula (1), but also to the quantities $(\partial u_1/\partial x_3)^2$ and $(\partial u_1/\partial x_1)^2$, whose mean value (practically coinciding with the volume average over V_0), as is known, is equal to $\frac{2}{15}\frac{\bar{\varepsilon}}{\nu}$ and, respectively, $\frac{1}{15}\frac{\bar{\varepsilon}}{\nu}$ ([2]). Therefore formulas (9)–(10) may also be applied to the field $\varepsilon_1(\mathbf{x}, t)$. Thus, the experimental result (2) suggests that, apparently, $\mu \simeq 0.4$. According to (7)–(8) it follows from this that *the influence*

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of fluctuations of the energy dissipation ε *should lead to an increase of the exponent* $2/3$ *and a decrease of the exponent* $-5/3$ *in the "two-thirds" and "five-thirds" laws for the structural functions and the turbulence spectrum by an amount close to* 0.04. Reliable detection of corrections of this order to the exponents in the empirical laws is apparently still beyond the limits of what is possible in experimental investigations in the field of turbulence, but the achievements of recent years give reason to hope that the accuracy required here can be attained fairly soon.

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Note: Figure translations are in progress. See original paper for figures.

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