CLASSIFICATION OF NONOSCILLATORY CASES FOR THE EQUATION

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Abstract

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MATHEMATICS

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CLASSIFICATION OF NONOSCILLATORY CASES FOR THE EQUATION

$\ddot{x} + p(t)\dot{x} + q(t)x = 0$

WITH SIGN-DEFINITE $q(t)$

(Presented by Academician L. S. Pontryagin on 16 II 1966)

1. Consider the equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \qquad (-\infty < a \leq t < b \leq \infty), \tag{1}$$

where $p(t), q(t)$ are real and $q(t)$ does not change sign on $[a, b)$. For simplicity we assume the coefficients to be continuous on $[a, b)$ (with the usual reservations, everything remains valid for coefficients locally summable on $[a, b)$). If the nontrivial solutions of equation (1) have a finite number of zeros on $[a, b)$, then the solutions of (1) are said to be nonoscillatory (this is certainly the case when $q(t) \leq 0$).

Below, for the nonoscillatory case, a complete classification is given of the possible types of fundamental systems of solutions of equation (1), taking into account the following characteristics of the behavior of solutions as $t \to b$: convergence to zero, to a nonzero limit, to infinity, increase or decrease near b (in the nonoscillatory case every solution is monotone near b). For the equation $\ddot{x} + q(t)x = 0$ with $b = \infty$, such a classification was given in the works of E. Hille ([1]) and I. M. Sobol ([2,3]); there are two types of behavior of solutions for $q(t) \geq 0$ and two types for $q(t) \leq 0$, depending on the convergence or divergence of

$$\int^{\infty} tq(t)\, dt.$$

In the general case the situation proves to be more varied, since a number of essentially new asymptotics are added; in particular, cases appear corresponding to stability and asymptotic stability of the trivial solution, whereas for $p(t) \equiv 0$,

$b = \infty$, in the nonoscillatory case instability always occurs (there are unbounded solutions).

2. As a fundamental system $x_1(t), x_2(t)$ in what follows, a pair of linearly independent solutions of (1), positive near b, is chosen, and $x_2(t)$ is chosen to be the minimal solution, i.e. the unique (up to a constant factor) solution satisfying the condition (c lies to the right of the zeros of the solution)

$$\int_c^b \exp\left(-\int^t p(\tau)\,d\tau\right) x_2^{-2}(t)\,dt = \infty \qquad \left(\text{whence } \frac{x_2}{x_1} \downarrow 0 \text{ as } t \to b\right).$$

The choice of $x_1(t)$, in contrast to $x_2(t)$, is not unique up to a factor. Accordingly, the symbol $x_1 \uparrow\downarrow 1$ encountered below means that among the nonminimal solutions there are both those monotonically increasing and those monotonically decreasing to unity (and consequently, obviously, to any nonzero constant). Correspondingly, for example, the notation $x_1 \uparrow 1$ means that any nonminimal positive solution monotonically increases to a finite limit; the notations $x_1 \downarrow 1$, $x_1 \uparrow \infty$, $x_1 \downarrow 0$ have analogous meanings. All assertions about monotonicity hold, of course, only for t sufficiently close to b; monotonicity may be understood everywhere in the strict sense, provided only that $q(t) \neq 0$ near b.

A decisive role is played by the convergence or divergence of the integrals

$$J_1(p) = \int^b \exp\left(-\int^t p(\tau)\,d\tau\right) dt, \quad J_2 = (p, q) = \int^b q(t) \exp\left(\int^t p(\tau)\,d\tau\right) dt,$$

$$J_3(p, q) = \int^b q(t)\,dt \int_t^b \exp\left(\int_s^t p(\tau)\,d\tau\right) ds,$$

$$J_4(p, q) = \int^b q(t)\,dt \int^t \exp\left(\int_s^t p(\tau)\,d\tau\right) ds.$$

The unspecified limits of integration may be chosen arbitrarily in $[a, b)$, since their choice does not affect convergence (J_3 is considered only when $J_1 < \infty$, and J_4 only when $J_1 = \infty$).

3. The main result of the present paper may be represented by the following table, which determines the character of the solutions as $t \to b$, depending on J_1, J_2, J_3, J_4.

	J_1	$\lvert J_2\rvert$	$\lvert J_3\rvert$	$\lvert J_4\rvert$	A: $q(t) \geq 0$	B: $q(t) \leq 0$
1	$< \infty$	$< \infty$			$x_1 \uparrow\downarrow$ $1x_2 \downarrow 0$	$x_1 \uparrow\downarrow$ $1x_2 \downarrow 0$
2	$< \infty$	∞	$< \infty$		$x_1 \downarrow$ $1x_2 \downarrow 0$	$x \uparrow$ $1x_2 \downarrow 0$
3	$< \infty$		∞		$* \ x_1 \downarrow$ $0x_2 \downarrow 0$	$x_1 \uparrow$ $\infty x_2 \downarrow 0$
4	∞			$< \infty$	$x_1 \uparrow$ $\infty x_2 \uparrow 1$	$x_1 \uparrow$ $\infty x_2 \downarrow 1$
5	∞			∞	$* \ x_1 \uparrow$ $\infty x_2 \uparrow$ ∞	$x_1 \uparrow$ $\infty x_2 \downarrow 0$

It is not difficult to see that the table exhausts all possible cases (clearly, in case 1 $|J_3| < \infty$, and in case 3 $|J_2| = \infty$). In cases 3A and 5A (marked by an asterisk), the nonoscillation of the solutions must be stipulated as an additional condition; in the remaining cases it is a consequence of the other conditions (in particular, nonoscillation is obvious when $q(t) \leq 0$), and therefore need not be stipulated.

Since the behavior of the solutions in cases 3B and 5B is analogous, there are 5 different nonoscillatory types of behavior of solutions for $q(t) \geq 0$ and 4 different types for $q(t) \leq 0$; altogether, for sign-constant $q(t)$ we obtain 8 different nonoscillatory types of behavior of solutions (since 1A and 1B are analogous).

4. Instead of J_3, J_4, one could consider the integrals occurring in ([6],[7]):

$$J_3^*(p, q) = \int^b dt \int^t q(s) \exp\left(\int_t^s p(\tau)\, d\tau\right) ds,$$

$$J_4^*(p, q) = \int^b dt \int_t^b q(s) \exp\left(\int_t^s p(\tau)\, d\tau\right) ds.$$

Taking into account the constancy of sign of $q(t)$, one can verify by elementary means that, when $J_1 < \infty$, the convergence of J_3 is equivalent to the convergence of J_3^*, while when $J_1 = \infty$, the convergence of J_4 is equivalent to the convergence of J_2 and J_4^*. If one constructs the table for J_1, J_2, J_3^*, J_4^*, then rows 1–4 will remain without chang-

tion, and the 5th row splits into two, and the end of the table will look as follows:

	J_1	$\|J_2\|$	$\|J_3^*\|$	$\|J_4^*\|$	A: $q(t) \geqslant 0$	B: $q(t) \leqslant 0$
...	...	...	...	...	... ...	... ...
5	∞	$< \infty$		∞	$* \, x_1 \uparrow \infty x_2 \downarrow \infty$	$x_1 \uparrow \infty x_2 \downarrow 0$
6	∞	∞			$-$	

Case 6A is impossible for nonoscillatory equations: the relations $J_1 = J_2 = \infty$ imply oscillation of the solutions (this is also true for sign-variable $q(t)$). Thus the table becomes somewhat more complicated, but it contains some additional information.

5. A number of assertions can be made about the rates of growth and decrease of solutions, and also about the behavior of the first derivatives. We shall not dwell on this here; we note only that in cases 1, 2, 4, when one of the solutions tends to a nonzero constant, the rate of growth or decrease of the other is easily determined from the formulas $(c_1, c > 0)$

$$x_1(t) = c_1 x_2(t) \int^t \exp\left(-\int^s p(\tau)\, d\tau\right) x_2^{-2}(s)\, ds + c_2 x_2(t),$$

$$x_2(t) = c x_1(t) \int_t^b \exp\left(-\int^s p(\tau)\, d\tau\right) x_1^{-2}(s)\, ds.$$

6. The classification given, so far as we know, is presented here for the first time; however, a number of special cases were investigated earlier by various authors. Thus the important case $p(t) \equiv 0,\ b = \infty$, as has already been noted, was studied in the works of E. Hille and I. M. Sobol' ; this case is contained in rows 4 and 5 of the table (for $p(t) \equiv 0,\ b = \infty$ we obviously have

$$J_1 = \infty, \qquad J_4 = \int^\infty t q(t)\, dt).$$

A number of particular results for $b = \infty$, concerning the distinction between cases 1 and 2, on the one hand, and 3 on the other, were obtained in works 3. Opyal' s [4], N. G. Georgiu' s [6] (the formulations of this work contain several superfluous requirements), and the author's [7] (Theorems 2 and 3). The specific nature of case 1A was used in work [8], where, in particular, it was studied how a change in $p(t)$ affects nonoscillation; in this question the principal significance lies in the presence of increasing or decreasing positive solutions.

7. The assertions pertaining to cases 3A and 5A look less effective than the others, since they contain as a premise the condition of nonoscillation, for the verification of which, as is known, there are no fully general and effective methods. In applications it is natural to use these assertions in combination with some criteria for nonoscillation (excluding, of course, such criteria as $q(t) \leqslant 0$, or $J_1, J_3 < \infty$, or $J_1 = \infty,\ J_4 < \infty$). Other schemes for applying these assertions are also possible, not connected with invoking conditions of nonoscillation; an example of this kind is furnished by Theorem 7 of [(7)].

8. Some of the assertions presented can be generalized to the case of arbitrary (generally speaking, complex-valued) $q(t)$; however, in doing so they lose the finality that is characteristic of the case of real sign-constant $q(t)$. Example: in order that equation (1) possess a fundamental system of solutions of the form $x_1(t) = 1+$

$+o(1)$, $x_2(t) = o(1)$ as $t \to b$ are sufficient, and in the case of sign-constant $q(t)$ also necessary, that the conditions $J_1(p) < \infty$, $J_3^*(p, |q|) < \infty$ be satisfied. This theorem for $b = \infty$ was formulated in (7) (Theorem 3; J_3^* can obviously be replaced by J_3). Another result of the same character: in order that there exist solutions of equation (1) tending to nonzero limits as $t \to \infty$, it is sufficient, and in the case of sign-constant $q(t)$ also necessary, that one of the following conditions be satisfied: 1) $J_1(p) < \infty$, $J_3(p, |q|) < \infty$; 2) $J_1(p) = \infty$, $J_4(p, |q|) < \infty$. The necessary and sufficient character of these assertions in the case of sign-constant $q(t)$ is seen directly from the tables.

9. In the proof it is expedient (by multiplying by $\exp \int p(t)\, dt$) to reduce equation (1) to the form

$$\frac{d}{dt}\left(r(t)\frac{dx}{dt}\right) + f(t)x = 0 \qquad (-\infty < a \leq t < b \leq \infty) \tag{2}$$

($r \geq 0$; r^{-1}, f are locally summable on $[a, b)$), since $J_1, J_2, J_3, J_4, J_3^*, J_4^*$ for (2) pass respectively into the more compact expressions

$$I_1(r) = \int^b \frac{dt}{r(t)}, \qquad I_2(f) = \int^b f(t)\, dt, \qquad I_3(r, f) = \int^b f(t)\, dt \int_t^b \frac{ds}{r(s)},$$

$$I_4(r, f) = \int^b f(t)\, dt \int^t \frac{ds}{r(s)}, \qquad I_3^*(r, f) = \int^b \frac{dt}{r(t)} \int^t f(s)\, ds,$$

$$I_4^*(r, f) = \int^b \frac{dt}{r(t)} \int_t^b f(s)\, ds.$$

The following proposition plays an essential role in the proof. Alongside (2), consider the equation

$$\frac{d}{dt}\left(r(t)\frac{dy}{dt}\right) + g(t)y = 0 \qquad (a \le t < b). \tag{3}$$

Let $g(t)$ be real and let the solutions of (3) be nonoscillatory. We choose the fundamental system y_1, y_2 for (3) according to the previous rules ($y_1, y_2 > 0$ near b, y_2 minimal as $t \to b$); in assertions 2), 3) the same applies to the fundamental system x_1, x_2 for (2). The connection between the behavior of the solutions of (3), (2) as $t \to b$ is characterized, with the aid of the function $\varepsilon(t) = f(t) - g(t)$, as follows:

1) if the (complex-valued) $\varepsilon(t)$ satisfies the condition

$$I(\varepsilon) = \int^b y_1(t)y_2(t)|\varepsilon(t)|\,dt < \infty, \tag{4}$$

then (2) has a fundamental system x_1, x_2 such that

$$x_1(t) = y_1(t)[1 + o(1)], \qquad x_2(t) = y_2(t)[1 + o(1)] \qquad (t \to b), \tag{5}$$

$$\dot{x}_1(t) = \dot{y}_1(t)[1+o(1)]+o\left(\frac{1}{ry_2}\right), \qquad \dot{x}_2(t) = \dot{y}_2(t)[1+o(1)]+o\left(\frac{1}{ry_1}\right) \quad (t \to b);$$

2) if $\varepsilon(t) \ge 0$, $I(\varepsilon) = \infty$, and the solutions of (2) are nonoscillatory, then

$$\frac{x_1(t)}{y_1(t)} \downarrow 0, \qquad \frac{x_2(t)}{y_2(t)} \uparrow \infty \qquad (t \to b);$$

3) if $\varepsilon(t) \le 0$, $I(\varepsilon) = \infty$, then

$$\frac{x_1(t)}{y_1(t)} \uparrow \infty, \qquad \frac{x_2(t)}{y_2(t)} \downarrow 0 \qquad (t \to b).$$

Assertion 1) for $r(t) = 1$, $b = \infty$ was obtained by A. Halanay in (5). It is also proved there that, for sign-constant $\varepsilon(t)$, condition (4) is necessary for any of the relations (5), under some additional assumptions (assertions 2), 3) show, in particular, that these assumptions are superfluous).

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Note: Figure translations are in progress. See original paper for figures.