ON THE THEORY OF PARABOLIC POTENTIALS

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ON THE THEORY OF PARABOLIC POTENTIALS

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In the present paper the investigation of surface parabolic potentials is continued; theorems are proved on homeomorphisms established by fractional powers of a parabolic operator in special Hölder spaces of exponentially increasing functions. These results find various applications. Here we present the theorems obtained on their basis concerning solvability and well-posedness of general boundary-value problems for the half-space in the case of systems with variable coefficients in classes of functions analogous to those in which the Cauchy problem was studied ([1]).

1. Lemma on the kernel. In the theory of parabolic boundary-value problems, an important role is played by the following

Lemma (on the kernel). *If*

$$G(t,x) = t^{-(n-1+2b)/2b}\Omega\left(\frac{x}{t^{1/2b}}\right)$$

is, for $x_n > 0,\ t > 0$, *a solution of the parabolic system*

$$\Lambda u \equiv \frac{\partial u}{\partial t} - \sum_{|k|=2b} A_k D^k u = 0,$$

and $\Omega(z)$ *satisfies the inequality*

$$|D^m\Omega(z)| \le C_m \exp\left\{-c\sum_{s=1}^{n}|z_s|^q\right\}, \qquad q > 0$$

in the half-space $E_n^+\{z_n > 0,\ -\infty < z_s < \infty,\ s = 1, 2, \dots, n-1\}$, *then*

$$\int \Omega(z', 0)\, dz' = 0.$$

The proof is carried out with the aid of the method set forth in ([5]).

2. Theorem on the boundedness of an integral operator of surface parabolic-potential type. Consider the integral operator

$$u(t,x) = If = \int_0^t d\tau \int (t-\tau)^{(1-n-2b)/2b} \Omega\left(\frac{x'-\xi'}{(t-\tau)^{1/2b}}, \frac{x_n}{(t-\tau)^{1/2b}}\right) f(\tau,\xi')\, d\xi' \tag{1}$$

with a function $\Omega(z)$ possessing the properties:

$$1) \quad \int \Omega(z',0)\, dz' = 0;$$

$$2) \quad |\Omega(z)-\Omega(y)| \le C|z-y|^\gamma \left[\exp\left\{-c\sum_{s=1}^n |z_s|^q\right\} + \exp\left\{-c\sum_{s=1}^n |y_s|^q\right\}\right].$$

Precisely such are the potentials whose kernels are derivatives of order r_j of the j-th column of the half-space fundamental matrix of solutions ([1]).

Denote by $C^l_{k(t)}(\Pi_n)$ the space of functions $f(t,x)$, defined in $\Pi_n\{-\infty < x_s < \infty;\ s=1,2,\dots,n;\ 0 \le t \le T\}$, having $[l]$ derivatives with respect to x, for which the norm is finite

$$\|f\|^l_{k(t)} = |f|^l_{k(t)} + [f]^l_{k(t)} = \sup_{(t,x)\in\Pi_n} \left\{ \sum_{|k|\le[l]} \left|D^k_x f(t,x)\right| \exp\left[-k(t)\sum_{s=1}^n |x_s|^q\right]\right\}$$
$$+ \sup_{\substack{(t,x)\in\Pi_n \\ (\tau,y)\in\Pi_n}} \left\{ \sum_{|k|<[l]} \frac{\left|D^k_x f(t,x) - D^k_y f(\tau,y)\right|}{[d(t,x;\tau,y)]^{\{l\}} \left[\exp\left\{k(t)\sum_{s=1}^n |x_s|^q\right\} + \exp\left\{k(\tau)\sum_{s=1}^n |y_s|^q\right\}\right]}\right\};$$

$$k(t) = \frac{(c-\varepsilon)a}{\left[(c-\varepsilon)^{2b-1} - a^{2b-1}t\right]^{1/(2b-1)}}; \quad c \text{ is from property 2) of the function } \Omega(z),$$

$$0<\varepsilon<c; \quad a \text{ is an arbitrary positive number}; \quad 0 < T < \left(\frac{c-\varepsilon}{a}\right)^{2b-1};$$

$$d(t,x;\tau,y) = \sqrt{|t-\tau|^{1/b} + |x-y|^2}.$$

Theorem 1. If $f(0,x') = 0,\ f(t,x') \in C^{\alpha}_{k(t)}(\Pi_{n-1});\ 0 < \alpha < 1$, then $u(t,x) = If$ belongs to $C^{\alpha}_{k(t)}(\Pi^{+}_{n})$ and

$$[u]_\alpha \le c_1 [f]_\alpha,$$

where $c_1 = c_1(\alpha, \gamma, n, 2b, C, c)$.

Theorem 1 generalizes to the case of $\vec{2b}$-parabolic potentials, i.e. potentials and spaces in which each coordinate x_s has its own weight $1/2b_s$ with respect to the time coordinate t.

3. Fractional powers of parabolic operators. Define parabolic operators of fractional order and their inverses; they are a natural generalization to functions of many variables of the operations of fractional differentiation and integration in the sense of Liouville:

$$\mathcal{J}^{\alpha}_{n} f = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau \int G_0(t-\tau, x-\xi) f(\tau,\xi)\, d\xi,$$

$$\Lambda^{\alpha} f = \mathcal{J}^{-\alpha}_{n} f = \frac{1}{\Gamma(1-\alpha)} \Lambda \int_0^t (t-\tau)^{-\alpha}\, d\tau \int G_0(t-\tau, x-\xi) f(\tau,\xi)\, d\xi,$$

$0 < \alpha < 1$, $G_0(t,x)$ is the fundamental solution of the equation $\Lambda u = 0$.

These operators are studied in the space $C^{l}_{k(t)}(\Pi_n)$ and the space $C^{(l,\beta)}_{k(t)}(\Pi_n)$ of functions $f(t,x)$ having in Π_n $[l + 2b\beta]$ derivatives with respect to x, for which the norm is finite

$$\|f\|^{(l,\beta)}_{k(t)} = \|f\|^{l+2b\beta}_{k(t)} + \|t^{-\beta} f\|^{l}_{k(t)}.$$

Theorem 2*. The operator $\mathcal{J}^{\alpha}_{n}$ maps the space $C^{l}_{k(t)}(\Pi_n)$ into the space $C^{(l,\alpha)}_{k(t)}(\Pi_n)$ and is bounded, i.e.

$$\|\mathcal{J}^{\alpha}_{n} f\|^{(l,\alpha)}_{k(t)} \le M \|f\|^{l}_{k(t)}.$$

The operator Λ^{α} maps the space $C^{(l,\alpha)}_{k(t)}(\Pi_n)$ into the space $C^{(l)}_{k(t)}(\Pi_n)$ and is bounded. For any $f \in C^{l}_{k(t)}$, $\Lambda^{\alpha} \mathcal{J}^{\alpha}_{n} f = f$.

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* Using Lemma 1 and Theorems 1, 2 in the case $k(t) \equiv 0$, one of the authors, by means of the standard method set forth in ([5]), obtained Schauder estimates

for solutions of general boundary-value problems for Petrovskii-parabolic systems with variable coefficients in bounded cylindrical domains with smooth boundaries. These results were reported by him at the All-Union Conference in Dushanbe (April 1964). For more general parabolic systems, deeper results on a priori estimates and the correct solvability of boundary-value problems were announced by V. A. Solonnikov ([3]).

4. General boundary-value problems for a half-space.

Consider the problem of finding a solution of the parabolic Petrovskii system

$$L(u) \equiv \frac{\partial u}{\partial t} - \sum_{|k|=2b} A_k(t,x) D^k u - \sum_{|k|\le 2b-1} A_k(t,x) D^k u \equiv$$

$$\equiv \frac{\partial u}{\partial t} - A_0(t,x,D)u - A_1(t,x,D)u = f(t,x) \qquad (2)$$

in the domain Π_n^+, satisfying the following initial and boundary conditions:

$$u\big|_{t=0} = \varphi(x); \qquad \lim_{x_n\to+0} B_j(t,x,D)u =$$

$$= \lim_{x_n\to+0} \big[B_{j0}(t,x,D)u + B_{j1}(t,x,D)u\big] = \psi_j(t,x'), \qquad (3)$$

$$j = 1, 2, \dots, bN.$$

Under the assumption that condition P([1]) is fulfilled:

$$\left|\det \int_\Gamma B_0(\tau,\xi,\sigma)\, A_{0+}^{-1}(\tau,\xi,p,\sigma)\, M_1(\sigma_n)\, d\sigma_n\right| \ge \delta_1\big(|\sigma'|^2 + |p|^{1/b}\big)^{m/2}$$

for arbitrary real σ', $(\tau,\xi)\in\Pi_n^+$, $p = a_0 + ip_1$, $a_0 > 0$, $-\infty < p_1 < \infty$, for the solution of the posed problem a special fundamental matrix of solutions $\widetilde{E}$ is constructed, whose principal term is the kernel $\widetilde{G}$ with columns $\widetilde{G}_j = \mathcal{G}_{n-1}^{a_j}(G_j)$; $a_j = (2b-1-r_j)/2b$; G_j are the columns of the half-space fundamental matrix of solutions studied in ([1], [2]), and then the solution of the problem is sought in the form

$$u(t,x) = \int_0^t d\tau \int \widetilde{E}(t,\tau,x,\xi')\mu(\tau,\xi')\, d\xi' + W(t,x) + V(t,x),$$

$$W(t,x)=\int_0^t d\tau\int Z(t,\tau,x,\xi)f_1(\tau,\xi)\,d\xi;\qquad V(t,x)=\int Z(t,0,x,\xi)\varphi_1(\xi)\,d\xi;$$

f_1 and φ_1 are the extensions by Whitney of f and φ from Π_n^+ to Π_n and from E_n^+ to E_n, respectively. Then the density is chosen so as to satisfy the boundary conditions (3). In doing so, because $\widetilde{E}$, generally speaking, has a reduced singularity, one arrives at a system of Volterra integral equations of the first kind with respect to μ, which, by applying the operator Λ^{α_j}, is transformed into an equivalent system of integral equations of the second kind with a quasiregular kernel. The solution thereby obtained is studied in detail; the fact of an increase in its smoothness with increasing smoothness of the data of the problem is established, from which the following theorems are obtained on the solvability of problem (2), (3) in classes of rapidly increasing functions and estimates of its norm in terms of the corresponding norms of the functions f,φ, and ψ_j.

Theorem 3. If

$$f\in C_{K(t)}^{l-2b+\alpha}(\Pi_n^+);\quad \varphi\in C_{K(t)}^{l+\alpha}(E_n^+),\quad \psi_j(t,x')\in C_{K(t)}^{l-r_j+\alpha}(\Pi_{n-1})$$

and the usual compatibility conditions are fulfilled, then for any $l\geq 2b$ and any $0<\alpha<1$ there exists a solution of problem (2), (3), belonging to $C_{K(t)}^{l+\alpha}(\Pi_n^+)$, and the estimate holds:

$$\|u\|_{K(t)}^{l+\alpha}\leq C\left(\|\varphi\|_{K(t)}^{l+\alpha}+\|f\|_{K(t)}^{l-2b+\alpha}+\sum_{j=1}^{bN}\|\psi_j\|_{K(t)}^{l-r_j+\alpha}\right).$$

5. Establishing uniqueness by methods of potential theory requires that the adjoint problem also turn out to be differential. Generalizing M. Schechter' s construction ([7]), we call a system of matrix $(N\times N)$ boundary operators $B_j(D)$, $j=1,2,\dots,b$, normal if the following conditions are satisfied: the orders r_j of the operators B_j are distinct and $\det B_{j_0}(\sigma)\neq 0$; B_{j_0} is the principal part of B_j. If the system of boundary operators is normal, then a system of boundary operators for the adjoint problem is constructed, which also turns out to be normal. If condition P is satisfied for the original problem, then, under the assumption $r_k+r_l\neq 2b-1,\ k\neq l$, it is also satisfied for the adjoint problem. Under this assumption, problem (2), (3) has a unique solution in any of the spaces $C_{k(t)}^{l+\alpha}(\Pi_n^+)$, and consequently, by Theorem 3, problem (2), (3) is correctly solvable in these spaces.

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Note: Figure translations are in progress. See original paper for figures.