APPLICATION OF THE AREA PRINCIPLE TO PROBLEMS ON NONOVERLAPPING FINITELY CONNECTED DOMAINS

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Abstract

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APPLICATION OF THE AREA PRINCIPLE TO PROBLEMS ON NONOVERLAPPING FINITELY CONNECTED DOMAINS

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In the author's paper ([1]) one generalized area theorem (namely the theorem of § 1) was proved for nonoverlapping simply connected domains, and examples were given of the application of this theorem to obtaining inequalities of a certain kind in some classes of functions univalent in the disk. The present article is devoted to extending the method of ([1]) to the case of nonoverlapping finitely connected domains.

Let $B^j,\ j=0,1,\dots,n;\ n=0,1,\dots,$ be arbitrary, henceforth fixed, finitely connected domains of the z-plane, with the complement of the domain B^j having no isolated points; let α_j be a fixed point of B^j; let $a_j,\ j=0,1,\dots,n$, be distinct fixed points of the w-plane and $a_0=\infty$; let $w=f_j(z),\ f_j(\alpha_j)=a_j$, be a function meromorphic, if $j=0$, and regular, if $j\neq 0$, in the domain B^j, univalently mapping B^j onto a domain D^j of the w-plane, and suppose that the domains D^j have no common points pairwise (nonoverlapping domains). Thus there is a system $\{f_j(z)\}_0^n$ of $n+1$ functions. The set of such systems of functions will be called the class $\mathfrak{M}$.

Let $g_j(w,t)$ be the Green' s function of the domain D^j, and

$$L_\rho^j=\{w:\ g_j(w,a_j)=\ln\rho\},\qquad D_\rho^j=\{w:\ g_j(w,a_j)>\ln\rho\},$$

$$D_{\rho\rho'}^j=\{w:\ \ln\rho\le g_j(w,a_j)<\ln\rho'\},\qquad 1<\rho<\rho'<\infty.$$

The complement G_ρ of $\bigcup_{j=0}^n G_\rho^j$, obviously, consists of a finite number of finitely connected domains $G_\rho^k,\ k=0,1,\dots,m$, each of which, for ρ sufficiently close to one, has a boundary Λ_ρ^k consisting of a finite number of simple closed analytic curves. In what follows we assume that the curves L_ρ^j and Λ_ρ^k are oriented so that the domains D_ρ^j and G_ρ^k lie on the left when moving in the direction of the corresponding curve (positive direction of traversal). We denote the area of G_ρ by $S(\rho)$ and put $S(1) = \lim_{\rho\to 1+0} S(\rho)$.

For an arbitrary domain B functions $P(z,a,b)$ and $Q(z,a,b)$, $z \in B$, $a \in B$ and $b \in B$, are introduced (see ([2]), Chap. VI). The corresponding functions for the domain G^j_ρ (and B^j) will be denoted by $P_{\rho j}(w,t,a_j)$ and $Q_{\rho j}(w,t,a_j)$ (respectively by $p_j(z,\zeta,\alpha_j)$ and $q_j(z,\zeta,\alpha_j)$). It is necessary to keep in mind the following properties of these functions. Let L^j_ρ consist of m_j closed curves L^{jk}_ρ, $k = 1,2,\dots,m_j$. Then for $w \in L^{jk}_\rho$ we have $P_{\rho j}(w,t,a_j) = -Q_{\rho j}(w,t,a_j) + C_{\rho jk}(t,a_j)$, where $C_{\rho jk}(t,a_j)$ does not depend on w. The function

$$Q_{\rho j}(w,t,a_j) = \ln \frac{w-t}{(w-a_j)(t-a_j)} + R_{\rho j}(w,t,a_j) + C_{\rho j},$$

where $C_{\rho j}$ is a constant, and the functions $P_{\rho j}(w,t,a_j)$ and $R_{\rho j}(w,t,a_j)$, regular in w on $\overline{D^j_\rho}$ for $t \in \overline{D^j_\rho}$, vanish if either $w = a_j$ or $t = a_j$, and

$$P_{\rho j}(w,t,a_j) = P_{\rho j}(t,w,a_j), \qquad R_{\rho j}(w,t,a_j) = R_{\rho j}(t,w,a_j).$$

Moreover, for every function $f(w)$ regular on $\overline{D^j_\rho}$, the following formula holds (an element of area in the t-plane):

$$f(w) - f(a_i) = \frac{1}{\pi} \iint_{D^j_\rho} f'(t)\, \frac{\partial}{\partial t} \overline{P_{\rho j}(t,w,a_j)}\, d\sigma_t. \tag{1}$$

Let $\{f_j(z)\}_0^n \in \mathfrak{M}$, and let a function $Q(w)$ be given, regular (and single-valued) in the domain G_{ρ_0}, $1 < \rho_0 < \infty$, with $Q'(w) \neq 0$ for $w \in G^k_{\rho_0}$, $k = 0,1,\dots,\overline{m}$. We may regard ρ_0 as such that every domain $D^j_{\rho_0}$ has the same connectivity as D^j. Introduce the functions

$$Q_{\rho j}(w) = -\frac{1}{2\pi i} \int_{L^j_{\rho'}} Q(w)\, \frac{\partial}{\partial t} Q_{\rho j}(t,w,a_j)\, dt, \qquad \rho < \rho' < \rho_0, \quad w \in D^j_{\rho\rho'},$$

$$R_{\rho j}(w) = \frac{1}{2\pi i} \int_{L^j_\rho} Q(w)\, \frac{\partial}{\partial t} Q_{\rho j}(t,w,a_j)\, dt = Q(w) - Q_{\rho j}(w), \qquad w \in D^j_\rho,$$

$$P_{\rho j}(w) = -\frac{1}{2\pi i} \int_{L^j_\rho} Q(t)\, \frac{\partial}{\partial t} P_{\rho j}(t,w,a_j)\, dt, \qquad w \in D^j_\rho.$$

It is easy to show that the functions $R_{\rho j}(w)$ and $P_{\rho j}(w)$ are regular in $\overline{D_\rho^j}$; for every function $f(w)$, regular in $\overline{D_\rho^j}$,

$$\int_{L_\rho^j} \overline{f(w)}\, Q'_{\rho j}(w)\, dw = 0;$$

for $w \in L_\rho^{jk}$ we have

$$P_{\rho j}(w) = -\overline{Q_{\rho j}(w)} + C'_{\rho jk},$$

where $C'_{\rho jk}$ is a constant; as $\rho \to 1+0$, the functions $R_{\rho j}(w)$ and $P_{\rho j}(w)$ tend uniformly inside D^j to certain functions $R_j(w)$ and $P_j(w)$ regular in D^j. Put $r_j(z) = R_j(f_j(z))$ and $p_j(z) = P_j(f_j(z))$.

Theorem 1. The inequality

$$\sum_{j=0}^{n} \iint_{B^j} |r'_j(z)|^2\, d\sigma_z \le \sum_{j=0}^{n} \iint_{B^j} |p'_j(z)|^2\, d\sigma_z \tag{2}$$

holds ($d\sigma_z$ is the element of area in the z-plane). Equality holds if and only if $S(1) = 0$.

Proof. Consider the expression for the area of the image of G_ρ under the mapping by the function $\xi = Q(w)$,

$$\sigma(\rho) = \iint_{G_\rho} |Q'(w)|^2\, d\sigma_w = \sum_{k=0}^{\overline{m}} \iint_{G_\rho^k} |Q'(w)|^2\, d\sigma_w > 0, \qquad 1 < \rho < \rho_0.$$

Using Green' s formula (see (2), p. 246), we obtain

$$\sigma(\rho) = \sum_{k=0}^{\overline{m}} \frac{1}{2i} \int_{\Lambda_\rho^k} \overline{Q(w)}\, Q'(w)\, dw = -\sum_{j=0}^{n} \frac{1}{2i} \int_{L_\rho^j} \overline{Q(w)}\, Q'(w)\, dw.$$

Hence, using the representation of the function $Q(w)$, the properties of the functions $R_{\rho j}(w)$, $Q_{\rho j}(w)$, and $P_{\rho j}(w)$, and Green' s formula, we have

$$\sigma(\rho) = -\sum_{j=0}^{n} \iint_{D_\rho^j} |R'_{\rho j}(w)|^2\, d\sigma_w + \sum_{j=0}^{n} \iint_{D_\rho^j} |P'_{\rho j}(w)|^2\, d\sigma_w > 0.$$

Passing to the limit as $\rho \to 1+0$, we obtain the inequality

$$\sum_{j=0}^{n} \iint_{D^j} |R'_j(w)|^2\, d\sigma_w \le \sum_{j=0}^{n} \iint_{D^j} |P'_j(w)|^2\, d\sigma_w. \tag{3}$$

It can be shown that all the integrals entering this inequality are finite. Now putting in (3) $w = f_j(z)$ for $w \in D^j$, we obtain inequality (2). The assertion about the sign of the equality in (2) follows from the fact that $\sigma(1) = \lim_{\rho\to 1+0} \sigma(\rho) = 0$ if and only if $S(1) = 0$.

Remark. If B^j, $j = 0, 1, \dots, n$, is the disk $|z| < 1$ and $\alpha_j = 0$, then Theorem 1 becomes a special case of the theorem from [(1)], § 1, when $Q(w)$ is regular in G_{ρ_0}.

For any two functions $g(z)$ and $h(z)$ regular in B^j, set

$$(g, h)_{B^j} = \iint_{B^j} g(z)\overline{h(z)}\, d\sigma_z, \qquad \|g\|_{B^j} = \sqrt{(g, g)_{B^j}}.$$

Theorem 2. *Let $s_j(z)$ $(j = 0, 1, \dots, n)$ be an arbitrary function regular in B^j such that $\|s_j'\|_{B^j} < \infty$. The following inequalities, equivalent to (2), hold:*

$$\sum_{j=0}^{n} \frac{|(r_j', s_j')_{B^j}|^2}{\|s_j'\|^2_{B^j}} \leqslant \sum_{j=0}^{n} \left(\|p_j'\|^2_{B^j} - \|r_j' - \lambda_j s_j'\|^2_{B^j}\right), \qquad \lambda_j = \frac{(r_j', s_j')_{B^j}}{\|s_j'\|^2_{B^j}}; \tag{4}$$

$$\left(\sum_{j=0}^{n} |(r_j', s_j')_{B^j}|\right)^2 \leqslant \sum_{j=0}^{n} \left(\|p_j'\|^2_{B^j} - \|r_j' - \lambda_j s_j'\|^2_{B^j}\right) \sum_{j=0}^{n} \|s_j'\|^2_{B^j}, \tag{5}$$

$$\lambda_j = \sum_{j=0}^{n} |(r_j', s_j')_{B^j}| \left(\sum_{j=0}^{n} \|s_j'\|^2_{B^j}\right)^{-1} e^{i\theta_j}, \qquad \theta_j = \arg(r_j', s_j')_{B^j};$$

$$\left|\sum_{j=0}^{n} (r_j', s_j')_{B^j}\right|^2 \leqslant \sum_{j=0}^{n} \left(\|p_j'\|^2_{B^j} - \|r_j' - \lambda_j s_j'\|^2_{B^j}\right) \sum_{j=0}^{n} \|s_j'\|^2_{B^j}, \tag{6}$$

$$\lambda_j = \sum_{j=0}^{n} (r_j', s_j')_{B^j} \left(\sum_{j=0}^{n} \|s_j'\|^2_{B^j}\right)^{-1}.$$

In inequality (4) we assume that $s_j'(z) \not\equiv 0$ in B^j for $j = 0, 1, \dots, n$; in (5) and (6), that for at least one j the function $s_j'(z) \not\equiv 0$ in B^j.

Inequalities (4), (5), and (6) are obtained analogously to the way this is done in the paper [(1)], § 2,1°. In inequalities (4), (5), and (6) the terms $\|r_j' - \lambda_j s_j'\|_{B^j}$ may be omitted. In this case it is easy to indicate the conditions under which the equality sign will hold in the resulting inequalities.

In connection with the above, let us make several remarks:

1. It is possible not to introduce the functions $R_{\rho j}(w)$, $Q_{\rho j}(w)$, and $P_{\rho j}(w)$, but to use the expansion of $Q(w)$ into a series with respect to a Laurent system of functions for the domain D^j_ρ (see [3]), analogously to the way this is done in [4] for the case $n = 0$. In this case, instead of (2), we obtain series of squares of the moduli of the coefficients of the expansions indicated above.

2. As the function $Q(w)$, it is convenient to take various linear combinations of the functions $\ln(w - w_{0\nu})$, w^k, $\ln(w - w_{j\nu})/(w - a_j)$, $(w - w_{j\nu})^{-k}$, where ν and k are natural numbers and $w_{j\nu}$ $(j = 0, 1, \dots, n)$ are various points of D^j. For such a function $Q(w)$, the functions $R_j(w)$ and $P_j(w)$ are easily constructed. As the functions $s_j(z)$ one may take linear combinations of the functions

$$r_j(z, z'_{j\nu}, \alpha_j) \quad \text{and} \quad \frac{\partial^k}{\partial \xi^k}\, p_j(z, \xi, \alpha_j)\bigg|_{\xi = z'_{j\nu}},$$

where $z'_{j\nu}$ are various points of B^j (we assume here that $\infty \in B^j$). These functions possess the reproducing property [1].

Example. Let $z_{j\nu}$ and $z'_{j\nu}$ be arbitrary points of B^j; $\gamma_{j\nu}$ and $\gamma'_{j\nu}$ arbitrary numbers $(j = 0, 1, \dots, n;\ \nu = 1, 2, \dots, m)$;

$$Q(w) = \sum_{j=0}^{n} \sum_{\nu=1}^{m} \gamma_{j\nu} \psi_j(w, z_{j\nu}),$$

where $\psi_0(w, \zeta) = \ln(w - f_0(\zeta))$ and $\psi_j(w, \zeta) = \ln[(w - f_j(\zeta))(w - a_j)^{-1}]$ for $j = 1, 2, \dots, n$;

$$s_j(z) = \sum_{\nu=1}^{m} \gamma'_{j\nu} p_j(z, z'_{j\nu}, \alpha_j).$$

Then, for example, inequality (4) (without the terms $\|r'_j - \lambda_j s'_j\|_{B^j}$) takes the form

$$\sum_{j=0}^{n} \frac{\left| \sum_{k=0}^{n} \sum_{\nu,\nu'=1}^{m} \gamma_{k\nu} \gamma'_{j\nu'} \varphi_{kj}(z_{k\nu}, z'_{j\nu'}) \right|}{\sum_{\nu,\nu'=1}^{m} \gamma'_{j\nu} \overline{\gamma'_{j\nu'}}\, p_j(z'_{j\nu}, z'_{j\nu'}, \alpha_j)} \le \sum_{j=0}^{n} \sum_{\nu,\nu'=1}^{m} \gamma_{j\nu} \overline{\gamma_{j\nu'}}\, p_j(z_{j\nu}, z_{j\nu'}, \alpha_j),$$

$$\varphi_{kj}(\zeta,z)=\begin{cases}\ln\dfrac{(f_j(z)-f_k(\zeta))(a_k-a_j)}{(f_j(z)-a_k)(f_k(\zeta)-a_j)}, & j\neq k,\ j\neq 0,\ k\neq 0;\\ \ln\dfrac{f_0(\zeta)-f_j(z)}{f_0(\zeta)-a_j}, & j\neq 0,\ k=0;\\ \ln\dfrac{f_0(z)-f_k(\zeta)}{f_0(z)-a_k}, & j=0,\ k\neq 0;\\ \ln\dfrac{(f_k(z)-f_k(\zeta))f_k'(\alpha_k)}{(f_k(z)-a_k)(f_k(\zeta)-a_k)}+q_k(\zeta,z,\alpha_k), & j=k\neq 0;\\ \ln[(f_0(\zeta)-f_0(z))f_0'(\alpha_0)]+q_0(\zeta,z,\alpha_0), & j=k=0.\end{cases}$$

Here the branches of the function $\varphi_{kj}(\zeta,z)$ have been chosen for which $\varphi_{kj}(\zeta,a_j)=0$; by $f_0'(\alpha_0)$ we mean

$$\lim_{z\to\alpha_0}\frac{1}{(z-\alpha_0)f_0(z)}.$$

3. Inequality (2) may be replaced by the inequality

$$\sum_{j=0}^{n}\|r_j'-g_j\|_{B^j}^2\leq\sum_{j=0}^{n}(\|p_j'\|_{B^j}^2+\|g_j\|_{B^j}^2-2\operatorname{Re}(r_j',g_j)_{B^j}),$$

where $g_j(z)$ is a function regular in B^j such that $\|g_j'\|_{B^j}<\infty$. In this case inequalities (4), (5), and (6) are modified in an obvious way. As the function $s_j'(z)$, one may now take a linear combination of the functions

$$K_j(z,z_{j\nu}'),\qquad \frac{\partial^k}{\partial\bar\zeta^k}K_j(z,\zeta)\bigg|_{\zeta=z_{j\nu}'},$$

where $K_j(z,\zeta)$ is the reproducing kernel in the class $A(B^j)$ of functions $f(z)$, regular in B^j and such that $\|f\|_{B^j}<\infty$, i.e.

$$f(\zeta)=\iint_{B^j}K_j(z,\zeta)f(z)\,d\sigma_z,\qquad \zeta\in B^j,\quad f(z)\in A(B^j).$$

4. Analogously to the way this was done in (1), one may consider the case when $Q(w)$ is a function having a regular (and single-valued) derivative in G_{ρ_0}, $1<\rho_0<\infty$.

All that has been presented makes it possible to carry over many results of work (1) to the case of multiply connected domains and to obtain a number of other results.

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Note: Figure translations are in progress. See original paper for figures.