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Abstract

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MATHEMATICS

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ON A NECESSARY CONDITION FOR AN EXTREMUM IN CONTROL PROBLEMS WITH AFTEREFFECT

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Let U be a convex, bounded, weakly closed class of summable real r-dimensional vector-functions on $[0, T]$ ($T > 0$ fixed). The class U is called the **class of controls**, and a vector-function

$$u(t) = (u^1(t), \dots, u^r(t)) \in U$$

is called a **controlling vector-function** or a **control**. For example, as U one may consider classes of summable r-dimensional vector-functions on $[0, T]$ satisfying on $[0, T]$ one of the following constraints:

$$|u^i(t)| \le \alpha_i(t); \qquad \alpha_i(t) \ge 0; \qquad i = 1, \dots, r; \qquad t \in [0, T]; \tag{1}$$

$$u^*(t)N(t)u(t) \le \beta(t); \qquad \beta(t) \ge 0; \qquad t \in [0, T]; \tag{2}$$

$$\int_0^T u^{i2}(t)\, dt \le C_i; \qquad 0 < C_i < \infty; \qquad i = 1, \dots, r; \tag{3}$$

$$\int_0^T u^*(t)N(t)u(t)\, dt \le C; \qquad 0 < C < \infty, \tag{4}$$

where $\alpha_i(t)$ $(i = 1, \dots, r)$, $\beta(t)$ are piecewise continuous functions bounded on $[0, T]$; $N(t)$ is an $r \times r$ matrix, positive definite on $[0, T]$, with piecewise continuous entries bounded on $[0, T]$; the asterisk denotes transposition.

We consider the problem of finding the minimum on U of a certain functional $I(u)$, described below. A general problem of this kind was considered, in particular, in (1).

On $[0, T]$ a system of n differential equations with variable delay is given:

$$dx(t)/dt = \dot{x}(t) = f(x(t), x(t - h_1(t)), u(t), t); \tag{5}$$

$$x(0) = x_0(t) \qquad \text{for } t \in [-h_1(0), 0]. \tag{6}$$

The function $v(t) = t - h_1(t)$ is a strictly increasing continuously differentiable real function on $[0, T]$,

$$0 < h_1(t) < \infty \qquad \text{for } t \in [0, T]; \qquad \min_{t \in [0,T]} h_1(t) > 0.$$

Then there exists an inverse function $t = r_1(v)$, which is also a strictly increasing continuously differentiable real function on $[-h_1(0), T - h_1(T)]$; $u \in U$, $f(x, y, u, t) = (f^1, \dots, f^n)$ is an n-dimensional real vector-function, continuous in x^i, y^j, u^k, t and continuously differentiable in x^i, y^j, u^k $(i, j = 1, \dots, n;\ k = 1, \dots, r)$ in the domain of admissible values x^i, y^j, u^k, determined by the class of controls U, the system (5), (6), and the initial vector-function $x_0(t)$, given and continuous on $[-h_1(0), 0]$.

By $x(t, u) = (x^1, \dots, x^n)$ we denote the solution of system (5) with initial conditions (6).

A functional is given

$$I(u) = \int_0^T g(x(t, u), x(t - h_2(t), u), u(t), t)\, dt, \tag{7}$$

where $\nu(t) \equiv t - h_2(t)$ is a strictly increasing continuously differentiable real function on $[0, T]$; $0 \le h_2(t) < \infty$ for $t \in [0, T]$. Let $t = r_2(\nu)$ be the function inverse to $\nu(t)$ (it also is strictly increasing and continuously differentiable on $[-h_2(0), T - h_2(T)]$); $g(x, y, u, t)$ is a scalar real function continuous in x^i, y^j, u^k, t and continuously differentiable in x^i, y^j, u^k $(i, j = 1, \dots, n;\ k = 1, \dots, r)$ in the domain of admissible values of x^i, y^j, u^k and for $t \in [0, T]$.

We assume that if $h_2(0) > h_1(0)$, then $x(t)$ is prescribed and continuous on $[-h_2(0), -h_1(0)]$.

From the class of controls U it is required to find a control $u \in U$ such that

$$I(u) = \min_{v \in U} I(v). \tag{8}$$

Such a $u \in U$ is called an optimal control.

Theorem. *In order that the control $u \in U$ give the functional (7) its minimal value on U, it is necessary, and in the case of convexity of $I(u)$ also sufficient, that*

$$\min_{v \in U} \int_0^T \sum_{i=1}^{r} \left[\left(\frac{\partial f(\tau)}{\partial u^i} \right)^* \psi(\tau) + \frac{\partial g(\tau)}{\partial u^i} \right] (v^i(\tau) - u^i(\tau))\, d\tau = 0, \tag{9}$$

where $\psi(\tau)$ is an n-dimensional vector-function satisfying the system of differential equations

$$\dot{\psi}(\tau) = \begin{cases} -\left(\dfrac{\partial f(\tau)}{\partial x}\right)^{*} \psi(\tau) - \left(\dfrac{\partial f(r_1(\tau))}{\partial y}\right)^{*} \psi(r_1(\tau))\, \dot{r}_1(\tau) - C(\tau), & \text{for } \tau \in [0, T - h_1(T)], \\ -\left(\dfrac{\partial f(\tau)}{\partial x}\right)^{*} \psi(\tau) - C(\tau), & \text{for } \tau \in [T - h_1(T), T], \end{cases} \tag{10}$$

$$\psi(T) = 0; \tag{11}$$

$$f(\tau) = f(x(\tau, u), x(\tau - h_1(\tau), u), u(\tau), \tau);$$

$$g(\tau) = g(x(\tau, u), x(\tau - h_2(\tau), u), u(\tau), \tau);$$

$$\partial g/\partial x = (\partial g/\partial x^1, \dots, \partial g/\partial x^n); \qquad \partial g/\partial y = (\partial g/\partial y^1, \dots, \partial g/\partial y^n);$$

$$\partial f/\partial x = \begin{pmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \cdots & \cdots & \cdots \\ \partial f^n/\partial x^1 & \cdots & \partial f^n/\partial x^n \end{pmatrix}; \qquad \partial f/\partial y = \begin{pmatrix} \partial f^1/\partial y^1 & \cdots & \partial f^1/\partial y^n \\ \cdots & \cdots & \cdots \\ \partial f^n/\partial y^1 & \cdots & \partial f^n/\partial y^n \end{pmatrix};$$

$$C(\tau) = \begin{cases} \dfrac{\partial g(\tau)}{\partial x} + \dfrac{\partial g(r_2(\tau))}{\partial y}\, \dot{r}_2(\tau), & \text{for } \tau \in [0, T - h_2(T)], \\ \dfrac{\partial g(\tau)}{\partial x}, & \text{for } \tau \in [T - h_2(T), T]. \end{cases}$$

The proof of this theorem is carried out using the results obtained in (1).

If delay is absent both in system (5) and in functional (7) (i.e. $h_1(t) \equiv h_2(t) \equiv 0$), then the necessary condition (9), as is easy to see, is preserved, and the system for $\psi(\tau)$ instead of (10) will have the form

$$\dot{\psi}(\tau) = -(\partial f(\tau)/\partial x)^{*} \psi(\tau) - \partial g(\tau)/\partial x, \tag{12}$$

$$\psi(T) = 0. \tag{13}$$

It is clear that in this case, for classes of controls with constraints of the form (1) and (2), condition (9) is a "linearization" of L. S. Pontryagin' s "maximum principle" (^{2}). Verifying the fulfillment of condition (9) is simpler than verifying the fulfillment of the "maximum principle." Moreover, the necessary condition

(9) is valid for classes of controls U of a more general form than the "maximum principle" (in particular, for classes of controls with constraints of integral type (3), (4)).

Similarly, for the case of a constant delay

$$h_1(t) \equiv h_2(t) \equiv h = \text{const}$$

a necessary condition has been obtained which is (for classes of controls of the form (1), (2)) a linearization of the "maximum principle" (([2]), pp. 236–250). For the case of a variable delay (and classes of controls of the form (1), (2)), a necessary condition analogous to the "maximum principle" was obtained by Yu. F. Kazarinov ([3]).

A necessary condition analogous to (9) can be obtained for systems of integro-differential equations.

To find a control satisfying condition (9), one may apply one of the methods of successive approximations proposed in ([1]).

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