NONLINEAR THEORY OF THE DRIFT-CONE INSTABILITY

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Abstract

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NONLINEAR THEORY OF THE DRIFT-CONE INSTABILITY

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1. In the author' s work ([1]) it was shown that an inhomogeneous plasma confined in a magnetic field is unstable with respect to high-frequency, $\omega \gg \omega_{Hi}$, flute-type perturbations ($k_z = 0$), if the distribution of ions over transverse velocities is not Maxwellian and if, moreover, there is an interval of velocities for which $\partial f_0/\partial v_\perp > 0$. Here $\omega_{Hi} = eH/m_i c$ is the ion cyclotron frequency, $\mathbf{H} \parallel z$, $k_z = (\mathbf{kH})/H$ is the projection of the wave vector $\mathbf{k}$ along the magnetic field $\mathbf{H}$; f_0 is the ion distribution function over transverse velocities. This result was illustrated for the simplest form $f_0 \sim \delta(v_\perp^2 - v_0^2)$. Later Rosenbluth and Post ([2]) drew attention to the fact that a plasma with $\partial f_0/\partial v_\perp > 0$, as a rule, is realized in adiabatic traps owing to the presence of a loss cone. Therefore the instability mentioned, which may be called the drift-cone instability, can lead to an enhanced loss of particles from an adiabatic trap. The aim of the present work is to obtain estimates for the time of this loss.

2. In the linear approximation, oscillations of an inhomogeneous plasma with $k_z = 0$ and $\omega_{Hi} \ll \omega \ll \omega_{He}$ are described by the dispersion equation ([1]):

$$\varepsilon(\mathbf{k},\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{k_y \chi \omega_{pi}^2}{\omega_{Hi}\omega k^2} - \frac{\omega_{pi}^2}{nk^2} \int \left\{ \frac{\partial f_0}{\partial \varepsilon_\perp} \left(1 - \frac{\omega(1-\chi v_y/\omega_{Hi})}{\omega - \mathbf{kv}}\right) - \frac{\chi k_y}{m_i \omega_{Hi}} \frac{f_0}{\omega - \mathbf{kv}} \right\} d\mathbf{v} = 0. \quad (1)$$

Here $\varepsilon_\perp = v_\perp^2/2$, $\chi = \partial \ln n/\partial x$ (it is assumed that there is a density gradient n in the direction of the x-axis);

$$\omega_{pe}^2 = 4\pi e^2 n/m_e, \qquad \omega_{pi}^2 = \frac{m_e}{m_i}\omega_{pe}^2.$$

Substituting into this equation f_0 of the form

$$f_0 = \frac{m_i(T+T_1)}{T^2} e^{-m_i v_\perp{}^2/2T} \left(1 - e^{-m_i v_\perp{}^2/2T}{}_1\right) \quad (2)$$

(the parameter T has the meaning of temperature, while the quantity $\left(\frac{2}{m_i}\frac{TT_1}{T+T_1}\right)^{1/2} = v_c$ corresponds to the velocity scale below which $\partial f_0/\partial \varepsilon_\perp > 0$ ($T_1 \ll T$), i.e., is the characteristic velocity of the loss cone), and assuming $\omega \ll k_\perp \sqrt{T_1/m_i}$, we reduce it to the form

$$1 + \frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{k_y \chi \omega_{pi}^2}{\omega_{He}\omega k^2} - \frac{i\sqrt{\pi\omega}}{kv_c}\frac{\omega_{pi}^2}{k^2}\frac{m_i(T+T_1)}{T^2}\left[1 - \left(\frac{T_1}{T+T_1}\right)^{1/2}\right] = 0. \quad (3)$$

For $v_c/v_T > \rho/a$ $\big(v_T = \sqrt{2T/m_i},\ \rho = \sqrt{T/m_i}/\omega_{Hi}$, and for not too small a ratio ρ/a,

$$\left(\frac{\rho}{a}\right)^{3/2}\left(\frac{v_c}{v_T}\right)^{1/2} > \frac{\omega_{Hi}^2}{\omega_{pi}^2}\left(1 + \frac{\omega_{pe}^2}{\omega_{He}^2}\right) \quad (4)$$

equation (3) describes two branches of oscillations for which $\operatorname{Im}\omega \sim \operatorname{Re}\omega$. One of these branches is unstable, $\operatorname{Im}\omega > 0$. Its increment $\gamma = \operatorname{Im}\omega$ reaches a maximum at

$$k^2 \approx \frac{\omega_{pi}^2}{v_T^2\left(1 + \omega_{pe}^2/\omega_{He}^2\right)}\left(\frac{\rho}{a}\frac{v_T}{v_c}\right)^{1/2}, \quad (5)$$

and is, in order of magnitude, equal to

$$\gamma = \left(\frac{\rho}{a}\right)^{3/4}\left(\frac{v_c}{v_T}\right)^{1/4}\frac{\omega_{pi}}{\left(1 + \omega_{pe}^2/\omega_{He}^2\right)^{1/2}}. \quad (6)$$

The second branch, for these same values of k, has an increment of the same order.

3. Since the increment of the unstable oscillations is not small in comparison with the frequency, $\gamma \sim \omega$, the nonlinear interaction between them, leading to the establishment of a stationary state, should be described by means of some approximate system of equations of strong turbulence ([3]). For this purpose we shall use equations of the type given in § 6 of Ref. ([3]). This system of equations is obtained by taking into account the simplest irreducible diagrams and, in the limit $\gamma \ll \omega$, passes over into the equations of weak turbulence describing three-wave processes ([4]).

The approximate system of equations of strong turbulence in the form in which it is given in § 6 of Ref. ([3]) was obtained under the assumption that $\omega \ll (\omega_{Hi}, \omega_{He})$, whereas in the case of interest to us $\omega_{Hi} \ll \omega \ll \omega_{He}$. In other words, the difference is that in Ref. ([3]) the ions are assumed to be magnetized, whereas in the present case they are considered to move rectilinearly. This requires a new derivation of the formula for the perturbed ion distribution function. The terms in it that are linear in the field are calculated without difficulty. The nonlinear terms in f, because the oscillations have a small phase velocity, turn out to be smaller than the corresponding terms for the electrons, which, evidently, have their former form. For this reason we shall take into account in the wave equation only the part of the ion distribution function that is linear in the field.

Using these remarks and taking into account that $k_z = 0$, the approximate system of equations of strong turbulence from Ref. ([3]) for the problem of interest to us can be represented in the following form:

$$I(k) = \frac{2}{\left|G_0^{-1}(k) - \Sigma(k)\right|^2} \int |\Gamma(k, k_1)|^2 I(k_1) I(k - k_1)\, dk_1; \tag{7}$$

$$\Sigma(k) = 4 \int \frac{I(k_1)\Gamma(k, k_1)\Gamma(k - k_1, -k_1)}{G_0^{-1}(k - k_1) - \Sigma(k - k_1)}\, dk_1; \tag{8}$$

$$\sigma(k) = \left(\frac{c}{H}\right)^2 \int \frac{[\mathbf{k}, \mathbf{k}_1]_z^2\, I(k_1)\, dk_1}{\omega - \omega_1 - \sigma(k - k_1)}; \tag{9}$$

$$\chi(k) = k_y - \left(\frac{c}{H}\right)^2 \int \frac{[\mathbf{k}, \mathbf{k}_1]_z^2 I(k_1) \chi(-k_1)\, dk_1}{[\omega - \omega_1 - \sigma(k - k_1)]\, [-\omega_1 - \sigma(-k_1)]}; \tag{10}$$

where

$$G_0^{-1}(k) = 1 + \varepsilon_0^i(k) + \frac{\omega_{pe}^2}{\omega_{He}^2} + \frac{\varkappa \omega_{pe}^2}{\omega_{He} k^2} \frac{\chi(k)}{\omega - \sigma(k)}; \tag{11}$$

$$\Gamma(k, k_1) = \frac{ic}{2H} \frac{\varkappa \omega_{pe}^2}{\omega_{He} k^2} \frac{[\mathbf{k}, \mathbf{k}_1]_z}{\omega - \sigma(k)} \left(\frac{\chi(k - k_1)}{\omega - \omega_1 - \sigma(k - k_1)} - \frac{\chi(k_1)}{\omega_1 - \sigma(k_1)} \right); \tag{12}$$

$$I(k)\delta(k + k_1) = \langle \varphi(k) \varphi(k_1) \rangle. \tag{13}$$

The function $\chi(k)$ introduced here is related to the function $\Omega'(k)$ of Ref. ([3]) by the relation

$$\Omega'(k) = \omega - \sigma(k) - \frac{\varkappa T_e}{m_e \omega_{He}} \chi(k); \tag{14}$$

$k = (\mathbf{k}, \omega)$; φ is the potential of the electric field of the wave.

The system of equations (7)–(10) must be supplemented by equations for the slowly varying part of the ion and electron distribution functions (averaged over the pulsations). Since in the perturbed distribution function for the ions only the terms linear in the field were taken into account, their slow distribution function will be described by the quasilinear equation

$$\frac{\partial f_0}{\partial t} + \mathbf{v}\nabla f_0 + [\mathbf{v}, \vec{\omega}_{Hi}]\frac{\partial f_0}{\partial \mathbf{v}} = \frac{\pi e^2}{m_i^2}\int k_\alpha k_\beta I(k)\frac{\partial}{\partial v_\alpha}\delta(\omega - \mathbf{kv})\frac{\partial f_0}{\partial v_\beta}\, dk. \qquad (15)$$

Assuming $\partial f_0/\partial t \ll \omega_{Hi} f_0$, averaging over the angle in velocity space, and supposing that $\omega \ll kv$, from (15) we obtain:

$$\frac{\partial f_0}{\partial t} = \frac{1}{v}\frac{\partial}{\partial v}vD(v)\frac{\partial f_0}{\partial v} + \frac{\partial}{\partial x}D_{xv}\frac{\partial f_0}{\partial v}, \qquad (16)$$

where

$$D(v) = \frac{e^2}{m_i^2 v^3}\int \frac{\omega^2}{k} I(k)\, dk, \qquad (17)$$

$$D_{xv} = -\frac{e^2}{m_i^2 \omega_{Hi}}\frac{1}{v^2}\int \frac{\omega k_y}{k} I(k)\, dk. \qquad (18)$$

(For $\omega \simeq kv$, the integrands in (17), (18) should be multiplied by $[1 - (\omega/kv)^2]^{-1/2}$, and the integration over dk must be carried out over the region $\omega \leq kv$.)

The Vlasov equation for the electrons, averaged over the pulsations, can be obtained using the results of work (3). In the same approximations that were adopted in writing equations (7)–(10), this equation has the form (as in (16), averaging over angles in velocity space has been performed)

$$\frac{\partial f_0^{(e)}}{\partial t} = -\frac{c}{H}\frac{\partial}{\partial x}v_\perp\frac{\partial}{\partial v_\perp}\int k_y \,\mathrm{Im}\, P(k, v)\, dk, \qquad (19)$$

where

$$P(k,v) = \frac{1}{\omega - \sigma(k)} \Bigg\{ -\frac{c\chi f_0 \chi(k,v)}{H} I(k) + 2\int \gamma_3(k,k_1) I(k_1) I(k-k_1) \times \Gamma_3(-k,-k_1)\,dk_1 G(-k) + 4\int \gamma_3(k,k_1) I(k_1) G(k-k_1) \times \Gamma_3(k-k_1,-k_1)\,dk_1 I(k) \Bigg\}, \tag{20}$$

$$\gamma_3(k,k_1,v) = \frac{ic}{2H}\frac{cf_0^{(e)}\chi}{H}[\mathbf{k},\mathbf{k}_1]_z \left(\frac{\chi(k-k_1)}{\omega-\omega_1-\sigma(k-k_1)} - \frac{\chi(k_1)}{\omega_1-\sigma(k_1)} \right). \tag{21}$$

4. Using (16), (18) and (7)–(10), it is not difficult to verify that, in the mean, the quasineutrality of the plasma is not violated, and both kinds of charges diffuse along the inhomogeneity with the same velocity. The characteristic time of spatial diffusion τ_x can be estimated with the aid of equation (16)

$$\tau_x \simeq av/D_{xv}. \tag{22}$$

Let us compare this time with the estimate for the diffusion time in velocity space

$$\tau_v \simeq v^2/D(v). \tag{23}$$

Their ratio, in order of magnitude, is equal to

$$\frac{\tau_x}{\tau_v} \simeq \frac{\omega\omega_{Hi}}{k\chi v_T^2} \simeq \left(\frac{a}{\rho}\right)^{1/2} \left(\frac{v_c}{v_T}\right)^{1/2}. \tag{24}$$

Thus, ions diffuse more rapidly in velocity space than in ordinary space. Since the loss of ions into the cone is associated with velocity diffusion, τ_v is thereby the time scale for the lifetime of the plasma in the trap.

Now, using the system of equations (7)–(10), let us estimate the amplitude of the "established" oscillations. Assuming that the oscillations are localized in an interval k, ω of the order of the excitation interval, it is not difficult to see that equations (7)–(10) admit solutions with neither very large nor very small $I(k)$. All the equations of this system prove to be mutually consistent only if, in order of magnitude, the relation

$$\left(\frac{c}{H}\right)^2 \int I(k)\,dk \simeq \frac{\omega^2}{k^4}, \tag{25}$$

is satisfied, where ω and k are the effective values of the frequency and wave vector entering the problem (their expressions are given in point 2).

Using (17), (23), and (25), we find that the diffusion time in velocity space for particles with $v \lesssim v_c$, in order of magnitude, is equal to

$$\tau_{v_c} \simeq \frac{(kv)^5}{\omega^4 \omega_{Hi}^2} \simeq \frac{1}{\omega_{Hi}} \left(\frac{a}{\rho} \frac{v_T}{v_c} \right)^{5/2}. \tag{26}$$

This time will characterize the escape from the trap of particles with such velocities, if it does not exceed the time of free flight of these particles along the trap.

If v_c is maintained constant by some means, then the time of spatial diffusion will have the scale

$$\tau_r \simeq \left\{ \frac{1}{a} \frac{e^2}{m_i^2 \omega_{Hi}} \left(\overline{\frac{1}{v^3}} \right) \omega \int I(k)\, dk \right\}^{-1} \simeq \left(\frac{a}{\rho} \right)^3 \frac{v_T}{v_c} \frac{1}{\omega_{Hi}}, \tag{27}$$

so that the coefficient of spatial diffusion will be

$$D_r \simeq \frac{\rho^2}{a} \nu_c. \tag{28}$$

This expression coincides with the commonly used estimate $D_r \sim \gamma/k^2$ (4).

5. From the consideration carried out above it follows that, when an attempt is made to increase the plasma density in an adiabatic trap up to values for which the inequality (4) is satisfied, an anomalously rapid loss of particles will occur, with characteristic time $\tau = \min(\tau_{v_c}, \tau_r)$. As follows from (26), (28), this time is the greater, the larger the magnetic field and the mirror ratio (recall that $v_T/v_c \simeq (H_{\text{pr}}/H_0)^{1/2}$, where H_{pr} is the field in the mirrors and H_0 is that at the center of the trap).

It follows from criterion (4) that, for a sufficiently strong magnetic field, the instability considered disappears. However, as was shown by M. N. Rosenbluth, when the magnetic field is increased the plasma does not become absolutely stable, since instead of the drift-cone instability there may occur the instability of oblique waves ($k_\parallel \neq 0$), discussed by Rosenbluth and Post ([2]) and by Galeev ([5]).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.