ELASTIC CONSTANTS AND THERMAL EXPANSION OF CERTAIN BODIES WITH A NONHOMOGENEOUS REGULAR STRUCTURE

THEORY OF ELASTICITY

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

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ELASTIC CONSTANTS AND THERMAL EXPANSION OF CERTAIN BODIES WITH A NONHOMOGENEOUS REGULAR STRUCTURE

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A method is proposed for determining the elastic constants and coefficients of thermal expansion of nonhomogeneous bodies whose regular structure is formed by a doubly periodic system of n arbitrarily located hollow cylinders of different diameters. The space between the cylinders is filled with a medium.

1. Let E_a, ν_a, α_a and E_s, ν_s, α_s be the elastic constants and coefficients of thermal expansion of the filler and of the medium, $z = x_2 + ix_3$, $\theta = T - T_0$ the temperature, and Ω_k' and Ω_k the outer and inner lateral surfaces of cylinders with centers at the points $a_k + P$ ($P = m\omega_1 + n\omega_2$, $\omega_2 = \omega_1 be^{i\alpha}$, $m, n = 0, \pm 1, ...$) (Fig. 1). Functions referring to the filler are denoted by the subscript a, and to the medium by s; further,

Fig. 1

$$\langle \sigma_{ik} \rangle = F^{-1} \int_F dF \, \sigma_{ik}, \qquad \langle \varepsilon_{ik} \rangle = F^{-1} \int_F dF \, \varepsilon_{ik}, \qquad F = \omega_1^2 b \sin \alpha. \quad (1)$$

The relation between $\langle \sigma_{ik} \rangle$ and $\langle \varepsilon_{ik} \rangle$ on areas remote from local disturbances, owing to the plane of elastic symmetry $x_1 = \text{const}$, will be

$$\begin{aligned}
\langle\varepsilon_{11}\rangle &= X_{11}\langle\sigma_{11}\rangle + X_{12}\langle\sigma_{22}\rangle + X_{13}\langle\sigma_{33}\rangle + \cdots + X_{16}\langle\sigma_{23}\rangle + \beta_{11}\theta,\\
\langle\varepsilon_{22}\rangle &= X_{12}\langle\sigma_{11}\rangle + X_{22}\langle\sigma_{22}\rangle + X_{23}\langle\sigma_{33}\rangle + \cdots + X_{26}\langle\sigma_{23}\rangle + \beta_{22}\theta,\\
\langle\varepsilon_{33}\rangle &= X_{13}\langle\sigma_{11}\rangle + X_{23}\langle\sigma_{22}\rangle + X_{33}\langle\sigma_{33}\rangle + \cdots + X_{36}\langle\sigma_{23}\rangle + \beta_{33}\theta,\\
\langle\varepsilon_{31}\rangle &= \cdots\cdots\cdots\cdots\cdots X_{44}\langle\sigma_{31}\rangle + X_{45}\langle\sigma_{12}\rangle \cdots\cdots,\\
\langle\varepsilon_{12}\rangle &= \cdots\cdots\cdots\cdots\cdots X_{45}\langle\sigma_{31}\rangle + X_{55}\langle\sigma_{12}\rangle \cdots\cdots,\\
\langle\varepsilon_{23}\rangle &= X_{16}\langle\sigma_{11}\rangle + X_{26}\langle\sigma_{22}\rangle + X_{36}\langle\sigma_{33}\rangle + \cdots + X_{66}\langle\sigma_{23}\rangle + \beta_{23}\theta.
\end{aligned} \tag{2}$$

Under shear in the planes (x_1, x_2) and (x_1, x_3), the displacements $u_1(z)$ will be nonzero. Let Φ_s and $\Phi_{a,k}$ be holomorphic functions in s and s_k,

$$\sigma_{12} - i\sigma_{31} = 2G\Phi(z), \qquad u_1 = \operatorname{Re}\int \Phi(z)\,dz. \tag{3}$$

If $\Phi_s(z+\omega_j) = \Phi_s(z)$ $(j = 1, 2)$, then the boundary conditions expressing equality of displacements and stresses on the contact surface are neces-

must be satisfied on the n contours located in one cell,

$$L_{s,k}\Phi_s - L_{a,k}\Phi_{a,k}\big|_{\Omega_k} = 0, \qquad l_{a,k}\Phi_{a,k}\big|_{\Omega_k} = 0, \tag{4}$$

where $L_{s,k}$, $l_{a,k}$ are differential operators of the boundary conditions; further, from (1):

$$\Phi_s = c_0 - \sum_{k=1}^{n}\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!}\,c_{k,s}\zeta^{(s-1)}(z-a_k), \qquad \sum_{k=1}^{n} c_{k,1} = 0; \tag{5}$$

$$\Phi_{a,k} = \sum_{m=-\infty}^{\infty} a_{m,k}(z-a_k)^m.$$

On the surfaces of the body, the average stresses are prescribed as

$$\langle\sigma_{12}\rangle - i\langle\sigma_{31}\rangle = 2G_a\langle\Phi_{a,k}\rangle + 2G_s\langle\Phi_s\rangle. \tag{6}$$

The formulas for the elastic constants are obtained by comparing the displacements of the body with a nonhomogeneous structure with the displacements in the body (1) under shear ([2])

$$X_{44}\langle\sigma_{31}\rangle + X_{45}\langle\sigma_{12}\rangle = -2c_0'' + 2\sum_{k=1}^{n}\left\{\delta_2' c_{k,2}' + \left(\delta_1' - \frac{2\pi}{\omega_1^2 b\sin\alpha}\right)c_{k,2}''\right\},$$

$$c_{k,s} = c'_{k,s} + ic''_{k,s}, \tag{7}$$

$$X_{45}\langle\sigma_{31}\rangle + X_{55}\langle\sigma_{12}\rangle = 2c'_0 - 2\sum_{k=1}^{n}(\delta'_1 c'_{k,2} - \delta'_2 c''_{k,2}), \qquad \frac{2}{\omega_1}\zeta\left(\frac{\omega_1}{2}\right) = \delta'_1 + i\delta'_2.$$

2. In solving the problem of stretching the body by stresses $\langle\sigma_{11}\rangle = 1$, the constants $X_{11}, X_{12}, X_{13}, X_{16}$ are determined. The complex potentials satisfying the periodicity conditions of the stressed state are chosen in the form of the series (5) and

$$\Psi_s = d_0 - \sum_{k=1}^{n}\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!}\left\{d_{k,s}\zeta^{(s-1)}(z-a_k) - c_{k,s}\eta^{(s)}(z-a_k)\right\}, \tag{8}$$

$$\Psi_{a,k} = \sum_{m=-\infty}^{\infty} b_{k,m}(z-a_k)^m.$$

Here elliptic functions of the form ([1,2]) have been introduced,

$$\eta(z) = \sum_{m,n}{}' \overline{P}\left\{(z-P)^{-1} + z^3P^{-3} + zP^{-2}\right\}.$$

The arbitrary constants are determined from the boundary conditions

$$L_{s,k}(\Phi_s, \Psi_s) - L_{a,k}(\Phi_{a,k}, \Psi_{a,k})\big|_{\Omega'_k} = f_k(\vartheta), \qquad l_{a,k}(\Phi_{a,k}, \Psi_{a,k}) = 0 \tag{9}$$

and from the condition that the principal vector of the forces applied at the boundary of the elementary cell is zero,

$$\overline{d}_0\overline{\omega}_j - \sum_{k=1}^{n}\overline{d}_{k,2}\overline{\delta}_j = -2c_0\omega_j + \delta_j\sum_{k=1}^{n}c_{k,2} + \overline{\tau}_j\sum_{k=1}^{n}\overline{c}_{k,2} \quad (j=1,2). \tag{10}$$

The elastic constants are determined by the formulas

$$X_{11}^{-1} = E_a\sum_{k=1}^{n}\xi_k + E_s\eta + 8\nu_s(\nu_a - \nu_s)G_s\left\{\frac{\nu_a}{\nu_s}\sum_{k=1}^{n}\xi_k a'_{0,k} + \right.$$

$$\left. + \sum_{k=1}^{n}c_{k,1}\xi'_k\lambda_k - \sum_{k=1}^{n}\sum_{s=2}^{\infty}\frac{\xi'_k c'_{k,2s}}{2s-1}\alpha_{s-1,0} - \left(\delta'_1 - \frac{\pi}{F}\right)\sum_{k=1}^{n}c'_{k,2} + \delta'_2\sum_{k=1}^{n}c''_{k,2}\right\}. \tag{11}$$

Here the dominant terms are only the first two terms,

$$\nu_{21} = \nu_s + (\nu_s - \nu_a)(\chi_s + 1)\left(c_0' - \delta_1' \sum_{k=1}^{n} c_{k,2}' + \delta_2' \sum_{k=1}^{n} c_{k,2}''\right),$$

$$\nu_{61} = 2(\nu_s - \nu_a)\left\{(\chi_s + 3)c_0'' - (\chi_s + 1)\left(\delta_1' \sum_{k=1}^{n} c_{k,2}' + \delta_2' \sum_{k=1}^{n} c_{k,2}''\right)\right\},$$

$$\nu_{31} = \nu_{21} + (\nu_s - \nu_a)(\chi_s + 1)\frac{2\pi}{F} \sum_{k=1}^{n} c_{k,2}', \qquad \nu_{ik} = -\frac{X_{ki}}{X_{11}}.$$

3. The remaining constants in (1) are found by solving the problem of the plane strained state of the body ($\langle\varepsilon_{11}\rangle = 0$). The form of the solution and the boundary conditions are defined in (4), (8), (9), (10). Taking successively $\langle\sigma_{22}\rangle = 1$, $\langle\sigma_{33}\rangle = \langle\sigma_{23}\rangle = 0$, etc., we obtain three systems of algebraic equations from which c_k and $c_{k,s}$ are found.

From the first system, when $\langle\sigma_{22}\rangle = 1$, $\langle\sigma_{33}\rangle = \langle\sigma_{23}\rangle = 0$, we have

$$X_{22} = -\nu_{21}X_{12} + \frac{\chi_s + 1}{2G_s}\left\{\frac{1}{4} + c_0' - \delta_1' \sum_{k=1}^{n} c_{k,2}' + \delta_2' \sum_{k=1}^{n} c_{k,2}''\right\}, \qquad (12)$$

$$X_{23} = X_{22} - \frac{1}{2G_s} + \nu_{21}(X_{12} - X_{13}) + (\chi_s + 1)\frac{\pi}{G_{sF}} \sum_{k=1}^{n} c_{k,2}''.$$

From the second system, for $\langle\sigma_{33}\rangle = 1$, $\langle\sigma_{22}\rangle = \langle\sigma_{23}\rangle = 0$, it follows that

$$X_{33} = -\nu_{31}X_{13} + \frac{\chi_s + 1}{2G_s}\left\{\frac{1}{4} + c_0' + \delta_2' \sum_{k=1}^{n} c_{k,2}'' - \left(\delta_1' - \frac{2\pi}{F}\right) \sum_{k=1}^{n} c_{k,2}'\right\}, \qquad (13)$$

$$X_{36} = -\nu_{31}X_{16} + \frac{1}{G_s}\left\{(\chi_s + 3)c_0'' - (\chi_s + 1)\left(\delta_1' \sum_{k=1}^{n} c_{k,2}'' + \delta_2' \sum_{k=1}^{n} c_{k,2}'\right)\right\}.$$

For $\langle\sigma_{23}\rangle = 1$, from the third system we have

$$X_{26} = -\nu_{61}X_{12} + \frac{\chi_s + 1}{2G_s}\left\{c_0 - \delta_1 \sum_{k=1}^{n} c_{k,2}' + \delta_2' \sum_{k=1}^{n} c_{k,2}''\right\}, \qquad (14)$$

$$X_{66} = -\nu_{61} X_{16} + \frac{1}{G_s} \left\{ 1 + (\chi_s + 3) c_0'' - (\chi_s + 1) \left(\delta_1' \sum_{k=1}^{n} c_{k,2}'' + \delta_2' \sum_{k=1}^{n} c_{k,2}' \right) \right\}.$$

Here $\xi_k = \pi(\lambda_k^2 - \varepsilon_k^2)/F$, $\xi_k' = \pi\lambda_k^2/F$, $\chi = 3 - 4\nu$, $a_{i,k}$ are the coefficients of the expansion of $\zeta^{(k)}(z)$ in a Laurent series.

4. At an elevated temperature θ, owing to the redistribution of stresses, as one moves away from the edge of the body the transverse sections $x_1 = \text{const}$ remain plane; therefore

$$\langle \varepsilon_{11} \rangle = \alpha_s \theta + \langle \varepsilon_{11} \rangle_s = \alpha_a \theta + \langle \varepsilon_{11} \rangle_a. \tag{15}$$

The general solution in this case is composed of the solution of the problem of stretching the body by the stresses $\langle \sigma_{11} \rangle$, without taking into account the interaction between the filler and the medium, and of a solution that takes this interaction into account. The complex potentials have the form (5) and (8). The components of the thermal tensor

the expansion are found in the form

$$\begin{aligned}
\beta_{11} &= \alpha_s - (\alpha_s - \alpha_a) \frac{X_{11}}{\nu_s - \nu_a} \left\{ (1 + \nu_s) E_a \sum_{k=1}^{n} \xi_k - (1 + \nu_a) \left(X_{11}^{-1} - \eta E_s^{-1} \right) \right\}, \\
\beta_{22} &= \alpha_s + (\alpha_s - \beta_{11}) \nu_{21} - (\alpha_s - \alpha_a)(1 + \nu_a) \frac{\nu_s - \nu_{21}}{\nu_s - \nu_a}, \\
\beta_{33} &= \alpha_s + (\alpha_s - \beta_{11}) \nu_{31} - (\alpha_s - \alpha_a)(1 + \nu_a) \frac{\nu_s - \nu_{31}}{\nu_s - \nu_a}, \\
\beta_{23} &= \nu_{61} \left\{ \alpha_s - \beta_{11} + (\alpha_s - \alpha_a) \frac{1 + \nu_a}{\nu_s - \nu_a} \right\}.
\end{aligned} \tag{16}$$

From the equations presented it is not difficult to obtain formulas for the constants in the case of simpler types of structures–orthorhombic, tetragonal, or hexagonal.

The formulas found are suitable for determining the physico-mechanical characteristics and the stressed state of synthetic materials of the glass-plastic type, in which the binder has viscoelastic properties; one need only replace the constants E_s, ν_s by the corresponding linear operators.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.