ON MULTIPLICATIVE REPRESENTATIONS OF BOUNDED ANALYTIC OPERATOR-FUNCTIONS

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Abstract

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MATHEMATICS

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ON MULTIPLICATIVE REPRESENTATIONS OF BOUNDED ANALYTIC OPERATOR-FUNCTIONS

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1. Consider the class $\mathfrak{B}$, consisting of operator-functions $X(\zeta)$, regular for $|\zeta| < 1$, with values in a separable Hilbert space $\mathfrak{H}$, for each of which: 1) $X^*(\zeta)X(\zeta) \leq I$ for $|\zeta| < 1$; 2) $X(0)$ is boundedly invertible; 3) $X(0) - I \in \mathfrak{S}_1$, where $\mathfrak{S}_1$ is the set of operators A with finite trace norm: $|A|_1 = \mathrm{sp}(A^*A)^{1/2} < \infty$ (1).

Theorem 1. *In order that an operator-function $X(\zeta)$ belong to the class $\mathfrak{B}$, it is necessary and sufficient that the representation*

$$X(\zeta) = \int_0^l \exp\{k[\zeta, \vartheta(t)]\, dE(t)\}\, U\Pi(\zeta),$$

$$\Pi(\zeta) = \prod_{j=1}^{r} \left(\frac{\zeta_j - \zeta}{1 - \bar{\zeta}_j \zeta} \frac{|\zeta_j|}{\zeta_j} P_j + Q_j \right), \tag{1}$$

hold, in which $r \leq \infty$; P_j are orthoprojections, $\dim P_j\mathfrak{H} = m_j < \infty$; $Q_j = I - P_j$; $|\zeta_j| < 1$; $\sum_{j=1}^{r} m_j(1 - |\zeta_j|) < \infty$; $\vartheta(t)$ is a nondecreasing scalar function $(0 \leq \vartheta(t) \leq 2\pi)$; $E(t)$ is a Hermitian increasing operator-function satisfying the condition $\mathrm{sp}\, E(t) \equiv t$; U is a unitary operator $(I - U \in \mathfrak{S}_1)$; $k(\zeta, \vartheta) = (\zeta + e^{i\vartheta})/(\zeta - e^{i\vartheta})$. Moreover, the partial products converge to the Blaschke-Potapov product $\Pi(\zeta)$ in the trace norm, and the integral products also converge to the multiplicative integral.

The last circumstance (which remained unnoted in (4), where convergence in the **uniform** norm was considered) makes it possible, in studying the expansion (1), to use the apparatus of determinants of infinite order (1).

Lemma. *If $X_j(\zeta) \in \mathfrak{B}$ $(j = 1, 2)$, then there exist such $Y_j(\zeta) \in \mathfrak{B}$ that*

$$X_1(\zeta)X_2(\zeta) = Y_2(\zeta)Y_1(\zeta), \qquad \det X_j(\zeta) = \det Y_j(\zeta).$$

This proposition, formulated for the finite-dimensional case in ([5]), is obtained here by using the possibility of approximating, in the metric $\mathfrak{S}_1$, a function $X(\zeta) \in \mathfrak{B}$ by finite Blaschke products, and also with the aid of the following compactness criterion:

If $A_n, B_n, C_n \in \mathfrak{S}_1$ $(n = 1, 2, ...)$, $A_n \geq 0$, $B_n \geq 0$, and for any $f, g \in \mathfrak{H}$

$$|(C_{nf}, g)|^2 \leq (A_{nf}, f)(B_{ng}, g),$$

then from the compactness in $\mathfrak{S}_1$ of the families $\{A_n\}$ and $\{B_n\}$ there follows the compactness of the family $\{C_n\}$ in $\mathfrak{S}_1$ (cf. ([4])).

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* An analogous proposition, established for $\dim \mathfrak{H} < \infty$ by V. P. Potapov ([2,3]) and partially extended to the infinite-dimensional case in ([4]), is also valid for operator-functions of the broader class $\mathfrak{B}$ $(J^* = J,\ J^2 = I)$, for which 1) is replaced by the condition $Y^*(\zeta)JY(\zeta) \leq J$.

A nondecreasing function $\vartheta(t)$ $(0 \leq t \leq l)$ will be called **canonical** if $0 \leq \vartheta(t) \leq 2\pi$, $\vartheta(t-0) = \vartheta(t)$, and if it assumes the value 2π at no more than one point of the segment $[0, l]$.

Theorem 2. *A function $X(\zeta) \in \mathfrak{B}$ admits a representation (1) with a canonical function $\vartheta(t)$, which (as well as the number $l \geq 0$) is uniquely determined by $X(\zeta)$. If t_0 is a point of increase of $\vartheta(t)$, then the operator $E(t_0)$ is also uniquely determined. The Blaschke–Potapov product is determined by $X(\zeta)$ up to a constant left unitary factor.*

The first assertion of the theorem follows from Theorem 1, the lemma, and the uniqueness of the multiplicative representation for the function $\det X(\zeta)$; the second follows directly from the general Theorem 3 given below. As for the third assertion, it is proved exactly as was done by V. P. Potapov ([3]) in the finite-dimensional case. Simple examples show that if t_0 is a point of constancy of $\vartheta(t)$, then $E(t_0)$ is, generally speaking, not determined uniquely by $X(\zeta)$.

Theorem 3. *Let*

$$X_j(t; \zeta) = \int_0^t \exp\{k[\zeta, \vartheta(\tau)]\}\, dE_j(\tau)\, U_j \qquad (j = 1, 2;\ 0 \leq t \leq l,\ |\zeta| < 1),$$

where $\vartheta(t)$ is a canonical function; U_j are unitary operators; $E_j(t)$ $(0 \leq t \leq l,\ E_j(0) = 0)$ are Hermitian operator-functions, continuous and of bounded variation in the uniform operator metric. Then, if

$$X_1(l; \zeta) \equiv X_2(l; \zeta) \quad (|\zeta| < 1),$$

then for every point of increase t_0 of the function $\vartheta(t)$ the equalities

$$X_1(t_0;\zeta) \equiv X_2(t_0;\zeta) \quad (|\zeta|<1), \qquad E_1(t_0) = E_2(t_0)$$

hold.

In the proof, theorems of the Phragmén-Lindelöf type are used, as well as infinite-dimensional analogues of the delicate results of V. P. Potapov ([3]) on the reconstruction of a multiplicative integral with variable upper limit t from its modulus $R(t)$.

2. Let $X(\zeta) \in \mathfrak{B}$,

$$\det X(\zeta) = e^{ia} \prod_{j=1}^{r} \frac{\zeta_j - \zeta}{1-\bar{\zeta}_j\zeta} \frac{|\zeta_j|}{\zeta} \exp\left\{ \int_0^{2\pi} k(\zeta,\vartheta)\, d\sigma(\vartheta) \right\}$$

$$(\operatorname{Im} a = \operatorname{Im} \sigma(\vartheta) = 0, \quad \vartheta \in [0,2\pi]).$$

If K is the closed unit disk and C is the unit circle, then we shall call a Borel set $M(\subset K)$ a **carrier** of the function $X(\zeta)$ if $\zeta_j \in M$ $(j=1,2,...)$ and the σ-measure on C is concentrated on $M \cap C$ $(\sigma(M\cap C) = \sigma(C))$.

A function $X_1(\zeta) \in \mathfrak{B}$ is called a (right) divisor of the function $X(\zeta) \in \mathfrak{B}$ if $X(\zeta)X_1^{-1}(\zeta) \in \mathfrak{B}$. In the set of all divisors of the function $X(\zeta)$ one naturally introduces the definitions of greatest common divisor (g.c.d.) and least common multiple (l.c.m.).

Theorem 4. *For every Borel set $M \subset K$, there exists a unique (up to a left unitary factor) divisor $X_M(\zeta)$ of the function $X(\zeta) \in \mathfrak{B}$, possessing the property that M and $K\setminus M$ are carriers of $X_M(\zeta)$ and $X(\zeta)X_M^{-1}(\zeta)$, respectively. Moreover, almost everywhere on the unit circle at least one of the operators $X_M(e^{i\vartheta})$ and $X(e^{i\vartheta})X_M^{-1}(e^{i\vartheta})$ is unitary. If $M_1, M_2, ...$ are Borel sets in K, then*

$$X_{\cap M_j}(\zeta) = \text{g.c.d.}\,\{X_{M_j}(\zeta)\}, \qquad X_{\cup M_j}(\zeta) = \text{l.c.m.}\,\{X_{M_j}(\zeta)\}.$$

The first assertion of the theorem in the case when M is an interval follows from the lemma and Theorem 3; then one considers an open set and, finally, an arbitrary Borel set M. After this, the proof of the third assertion presents no difficulty if one uses results, due to Yu. L. Shmul' yan ([6]), on the structure of a semigroup of bounded holomorphic scalar functions. As for the second assertion, in its proof the following fact is essentially used: if $X(\zeta) \in \mathfrak{B}$, then for almost all $\vartheta \in [0,2\pi]$

$$\lim_{r\to 1} X^*(re^{i\vartheta})X(re^{i\vartheta}) = X^*(e^{i\vartheta})X(e^{i\vartheta})$$

exists in the trace norm.* This makes it possible to use determinants of infinite order also for the study of boundary properties of functions of the class $\mathfrak{B}$.

The proof of the following proposition, partially formulated for the finite-dimensional case in ([5]), is based on Theorems 1, 3, and 4.

Theorem 5. *An operator-function $X(\zeta)$ belongs to the class $\mathfrak{B}$ if and only if it admits a representation convergent in the metric $\mathfrak{S}$*

$$X(\zeta) = \int_0^{2\pi} \exp\{k(\zeta,\vartheta)\, d\Sigma_a(\vartheta)\} \cdot \int_0^{2\pi} \exp\{k(\zeta,\vartheta)\, d\Sigma_s(\vartheta)\}\, U\Pi_1(\zeta)\Pi(\zeta), \quad (2)$$

$$\Pi_1(\zeta) = \prod_{j=1}^{r} \int_0^{l_j} \exp\{k(\zeta,\vartheta_j)\, dE_j(t)\}.$$

*Here $\Pi(\zeta)$ is a Blaschke–Potapov product; $r \le \infty$; $0 \le \vartheta_j < 2\pi$; $E_j(t)$ $(0 \le t \le l_j)$, $\Sigma_s(\vartheta)$, and $\Sigma_a(\vartheta)$ $(0 \le \vartheta \le 2\pi)$ are Hermitian nondecreasing operator-functions, with $\operatorname{sp} E_j(t) \equiv t$, while $\Sigma_s(\vartheta)$ and $\Sigma_a(\vartheta)$ $(\Sigma_s(0) = \Sigma_a(0) = 0)$ are, respectively, singular and absolutely continuous in the metric $\mathfrak{S}_1$; U $(I - U \in \mathfrak{S}_1)$ is a unitary operator. In addition, $\Sigma_s(\vartheta)$ is uniquely determined by $X(\zeta)$, and $\Sigma_a(\vartheta)$ by $X^*(e^{i\vartheta})X(e^{i\vartheta})$; the functions $\Pi(\zeta)$ and $\Pi_1(\zeta)$ are determined up to a left unitary factor by $X(\zeta)$. The function $X(\zeta)$ is inner (outer)** if and only if, in the representation (2), $\Sigma_a(\vartheta) \equiv 0$ (respectively, $\Pi(\zeta) \equiv \Pi_1(\zeta) \equiv I$, $\Sigma_s(\vartheta) \equiv 0$), i.e., if and only if $\det X(\zeta)$ is an inner (outer) scalar function.*

Theorems 4 and 5 can be extended to the matrix classes $A^{(n)}$, $D^{(n)}$, $H_\delta^{(n)}$ $(0 < \delta < \infty)$, which were the subject of study in ([5]), and also to some of their infinite-dimensional analogues.

3. The theorems presented can be used to study invariant subspaces of certain operators different from normal ones. Below a number of results of this kind are formulated.

Let T be a simple (in other terminology, completely nonunitary) contraction of a Hilbert space $\mathfrak{H}$ ([8,9a]), and suppose $\|T^{-1}\| < \infty$, $I - T^*T \in \mathfrak{S}_1$. It is not hard to see that one can construct the characteristic function $X(\zeta)$ of the operator T ([8,9b]), belonging to the class $\mathfrak{B}$. If M is an arbitrary Borel subset of K, then, as follows from Theorem 4 and the results of B. Sz.-Nagy and C. Foiaş ([9b], §§3, 4), there exist invariant subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of the operator T such that $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \{0\}$, $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2 = \mathfrak{H}$, and, if T_1 and T_2 are the operators induced by T on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then $X_M(\zeta)$ and $X_{K\setminus M}(\zeta)$ serve as the characteristic functions of the operators T_1 and T_2, respectively—

* As is known ([7]), a bounded operator function holomorphic in the unit disk has, almost everywhere on the unit circle, in general, only weak boundary values.

** A function $X(\zeta) \in \mathfrak{B}$ is called inner if the operator $X(e^{i\vartheta})$ is unitary for almost all $\vartheta \in [0, 2\pi]$, and outer if $X(\zeta)$ has no nonconstant inner divisors.

respectively (cf. (9^r), Theorem 8). In particular, if M is the interior of the unit disk, then $\mathfrak{H}_1$ is the closed linear span of the finite-dimensional invariant subspaces of the operator T, while the spectrum of the operator T_2 lies on the unit circle (a related result was obtained independently by M. S. Brodskii without using function-theoretic methods).

By choosing the set M in another way, one can arrange, for example, that the function $X_M(\zeta)$ be inner and $X_{K\setminus M}(\zeta)$ outer (see Theorem 5), i.e. that $T_1 \in C_{00}$, $T_2 \in C_{11}$ (9^r).

If some arc of the unit circle is free of the spectrum of the operator T, then, relying on a proposition due to M. S. Brodskii and Yu. L. Shmul' yan (([10]), Theorem 1), one can show that the subspace $\mathfrak{H}_1 = \mathfrak{H}_M$ is determined by the contraction T and by the set M uniquely; moreover, the relations

$$\bigcap \mathfrak{H}_{M_j} = \mathfrak{H}_{\cap M_j}, \qquad \overline{\sum \mathfrak{H}_{M_j}} = \mathfrak{H}_{\cup M_j}$$

hold for any collection $\{M_j\}$ of Borel subsets of the disk K.

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Note: Figure translations are in progress. See original paper for figures.