THE RELATEDNESS OF THREE BANACH SPACES

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Abstract

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MATHEMATICS

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THE RELATEDNESS OF THREE BANACH SPACES

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In this paper the symbol $E_1 \subset E_0$ is used to denote the Banach space E_1, normally embedded in the Banach space E_0 (see ([1])). Denote by W_α $(0 \leq \alpha \leq 1)$ the convex hull of the set

$$S_\alpha = \{x : \|x\|_{E_0}^{1-\alpha} \|x\|_{E_1}^{\alpha} \leq 1, \quad x \in E_1\}.$$

Definition 1. Let $E_0 \supset E \supset E_1$. The **index of the intermediate space** E with respect to the spaces E_0 and E_1 is the number

$$\alpha_i = \inf_{W_\alpha \subset S_1(E)} \alpha = \operatorname{ind}_{E_0,E_1}(E),$$

where $S_1(E)$ is the unit ball of the space E.

Definition 2. We shall say that a **continuous normal scale** (see ([2])) E_α $(0 \leq \alpha \leq 1)$ **connects the spaces** $E_0 \supset E \supset E_1$, if

$$E_{\alpha=0} = E_0, \qquad E_{\alpha=\operatorname{ind}_{E_0,E_1}(E)} = E, \qquad E_{\alpha=1} = E_1.$$

Definition 3. Three normally embedded Banach spaces $E_0 \supset E \supset E_1$ are called **related** if there exists a continuous normal scale E_α $(0 \leq \alpha \leq 1)$ connecting these spaces.

Consider the class $K(E_0, E_1)$ of continuous normal scales E_α $(0 \leq \alpha \leq 1)$ which connect the spaces $E_0 \supset E_1$ and majorize the minimal scale constructed from the spaces E_0 and E_1 (see ([3])).

Definition 4. By **absolutely related Banach spaces** E_0, E, E_1 we shall mean normally embedded spaces $E_0 \supset E \supset E_1$ for which any two normal scales $E_\alpha \in K(E_0, E)$ $(0 \leq \alpha \leq \operatorname{ind}_{E_0,E_1}(E))$ and $\hat{E}_\alpha \in K(E, E_1)$ $(\operatorname{ind}_{E_0,E_1}(E) \leq \alpha \leq 1)$, when glued at the point $\alpha = \operatorname{ind}_{E_0,E_1}(E)$, form a continuous normal scale (see ([1,2])).

Lemma 1. *In order that three normally embedded Banach spaces* $E_0 \supset E \supset E_1$ *be related, it is necessary and sufficient that, upon gluing the maximal scales* $E_\alpha^{\max}$ $(0 \le \alpha \le \alpha_i,\ E_{\alpha=\alpha_i} = E)$ *and* $\hat{E}_\alpha^{\max}$ $(\alpha_i \le \alpha \le 1,\ \hat{E}_{\alpha_i} = E,\ \hat{E}_{\alpha=1} = E_1)$ *at the point* $\alpha = \operatorname{ind}_{E_0,E_1}(E)$*, a continuous normal scale be obtained (see ([1,2])).*

Lemma 2. *Three normally embedded Banach spaces* $E_0 \supset E \supset E_1$ *are absolutely related if and only if the family of Banach spaces*

$$E_\alpha = \begin{cases} E_\alpha^{\min}(E_0, E); & 0 \le \alpha \le \alpha_i,\ E_{\alpha=\alpha_i} = E, \\ E_\alpha^{\min}(E, E_1); & \alpha_i \le \alpha \le 1, \end{cases}$$

where $E_\alpha^{\min}(E_0, E)$*,* $E_\alpha^{\min}(E, E_1)$ *are the minimal scales constructed from the spaces* $E_0 \supset E$ *and* $E \supset E_1$*, respectively, is a continuous normal scale (see ([3])).*

We shall formulate a criterion for the absolute relatedness of three normally embedded Banach spaces $E_0 \supset E \supset E_1$, restricting ourselves, for simplicity, to the case of a smooth unit sphere of the space E (see (4), p. 187, Definition 1).

Theorem 1. *In order that three normally embedded Banach spaces* $E_0 \supset E \supset E_1$ *be absolutely related, it is necessary and sufficient that, for all functionals* $f(x) \in E_0'$*, the inequality*

$$\|f\|_{E'} \ge \|f\|_{E_0'}^{1-\alpha_i} \|f\|_{E_1'}^{\alpha_i}. \tag{1}$$

Proof of sufficiency. On the basis of the definition of the norm of an element x in the minimal scale,

$$\|x\|_{E_\alpha^{\min}(E_0,E)} = \sup_{f \in E_0'} \frac{|f(x)|}{\|f\|_{E_0'}^{(\alpha_i-\alpha)/\alpha_i} \|f\|_{E'}^{\alpha/\alpha_i}} \qquad (x \in E_1,\ 0 \le \alpha \le \alpha_i),$$

$$\|x\|_{E_\alpha^{\min}(E,E_1)} = \sup_{f \in E'} \frac{|f(x)|}{\|f\|_{E'}^{(1-\alpha)/(1-\alpha_i)} \|f\|_{E_1'}^{(\alpha-\alpha_i)/(1-\alpha_i)}} \qquad (x \in E_1,\ \alpha_i \le \alpha \le 1).$$

Consider the function

$$\varphi(\alpha, x) = \begin{cases} \ln \|x\|_{E_\alpha^{\min}(E_0,E)}, & \text{if } 0 \le \alpha \le \alpha_i, \\ \ln \|x\|_{E_\alpha^{\min}(E,E_1)}, & \text{if } \alpha_i \le \alpha \le 1. \end{cases}$$

It is not hard to see that the absolute relatedness of the Banach spaces $E_0 \supset E \supset E_1$ is equivalent to the inequality

$$\varphi_\alpha'(\alpha_i - 0, x) \le \varphi_\alpha'(\alpha_i + 0, x), \qquad x \in E_1,$$

where $\varphi_\alpha'(\alpha_i - 0, x)$, $\varphi_\alpha'(\alpha_i + 0, x)$ are the left and right derivatives with respect to α of the function $\varphi(\alpha, x)$ at the point $\alpha = \alpha_i$.

Suppose that inequality (1) is satisfied for all elements $f \in E_0'$. By virtue of the smoothness of the space $E_0 \supset E$, we can choose a sequence of functionals $\{f_n\} \in E_0'$ such that

$$\lim_{n\to\infty} |f_n(x)|/\|f_n\|_{E'} = \|x\|_E \tag{2}$$

(see (2), Theorem 1).

Denote by

$$\Phi_n(\alpha, x) = \ln \frac{|f_n(x)|}{\|f_n\|_{E_0'}^{(\alpha_i-\alpha)/\alpha_i} \|f\|_{E'}^{\alpha/\alpha_i}} \qquad (0 \le \alpha \le \alpha_i,\ x \in E_1),$$

$$\Psi_n(\alpha, x) = \ln \frac{|f_n(x)|}{\|f_n\|_{E'}^{(1-\alpha)/(1-\alpha_i)} \|f_n\|_{E_1'}^{(\alpha-\alpha_i)/(1-\alpha_i)}} \qquad (\alpha_i \le \alpha \le 1,\ x \in E_1).$$

It is easy to see that

$$\varphi(\alpha, x) \ge \Phi_n(\alpha, x) \qquad (0 \le \alpha \le \alpha_i),$$

$$\varphi(\alpha, x) \ge \Psi_n(\alpha, x) \qquad (\alpha_i \le \alpha \le 1), \tag{3}$$

moreover, without loss of generality one may assume the existence of the limits

$$\lim_{n\to\infty} [\Phi_n(\alpha_i, x)]'_\alpha, \qquad \lim_{n\to\infty} [\Psi_n(\alpha_i, x)]'_\alpha.$$

From conditions (2) and (3) follow the inequalities

$$\varphi'_\alpha(\alpha_i - 0, x) \le \lim_{n\to\infty} [\Phi_n(\alpha_i, x)]'_\alpha,$$

$$\varphi'_\alpha(\alpha_i + 0, x) \ge \lim_{n\to\infty} [\Psi_n(\alpha_i, x)]'_\alpha.$$

Finally, by virtue of relation (1),

$$\frac{\|f_n\|_{E_0'}^{1/\alpha_i}}{\|f_n\|_{E_1'}^{1/\alpha_i}} = [\Phi_n(\alpha_i, x)]'_\alpha \le \frac{\|f_n\|_{E'}^{1/(1-\alpha_i)}}{\|f_n\|_{E_1'}^{1/(1-\alpha_i)}} = [\Psi_n(\alpha_i, x)]'_\alpha,$$

so that

$$\varphi'_\alpha(\alpha_i - 0, x) \le \varphi'_\alpha(\alpha_i + 0, x).$$

The sufficiency of the conditions of Theorem 1 is proved.

We proceed to present the scheme of the proof of necessity. First one proves

Lemma 3. *Let $E_0 \supset E_1$ be normally embedded Banach spaces, L a linear manifold of E_1, and L_{E_0}, L_{E_1} the linear manifold L endowed with the norms $\|x\|_{E_0}$ and $\|x\|_{E_1}$, respectively.*

Then for every element $x \in L$

$$\|x\|_{E_\alpha^{\min}(L_{E_0}, L_{E_1})} \geq \|x\|_{E_\alpha^{\min}(E_0, E_1)} \qquad (0 \leq \alpha \leq 1),$$

where $E_\alpha^{\min}(L_{E_0}, L_{E_1})$, $E_\alpha^{\min}(E_0, E_1)$ $(0 \leq \alpha \leq 1)$ are the minimal scales constructed from the spaces $L_{E_0} \supset L_{E_1}$ and $E_0 \supset E_1$.

Moreover, one can show that from the absolute relatedness of the finite-dimensional spaces $E_0 \supset E \supset E_1$ there follows inequality (1), under the condition that the unit sphere of the space E is smooth.

Let $f(x)$ be an arbitrary functional from the space E_0'. By assumption, the space E_1 is normally embedded in E and E_0; therefore, for any $\delta > 0$ there exist elements $x_0, x, x_1 \in E_1$ with norms $\|x_0\|_{E_0} = \|x\|_E = \|x_1\|_{E_1} = 1$, for which

$$|f(x_0)| = \|f\|_{E_0'} + \varepsilon_0, \qquad |f(x)| = \|f\|_{E'} + \varepsilon, \qquad |f(x_1)| = \|f\|_{E_1'} + \varepsilon_1, \quad (4)$$

where $\varepsilon_0, \varepsilon, \varepsilon_1 < \delta$.

Denote by L_{E_0}, L_E, L_{E_1} the subspaces of E_0, E, and E_1 generated by the elements x_0, x, x_1. Then

$$L_{E_0} \supset L_E \supset L_{E_1},$$

and the unit sphere of the space L_E is smooth. On the basis of Lemmas 2 and 3 we may assert that the Banach spaces L_{E_0}, L_E, L_{E_1} are absolutely related; consequently, the norm $\|\bar{f}\|_{L_E'}$ of the functional $\bar{f}(x)$, induced on L_{E_0} by the functional $f(x)$, satisfies the inequality

$$\|\bar{f}\|_{L_E'} \geq \|\bar{f}\|_{L_{E_0}'}^{1-\alpha_i} \|\bar{f}\|_{L_{E_1}'}^{\alpha_i}.$$

Taking into account the equalities (4), we obtain

$$\|\bar{f}\|_{L_{E_0}'} = \|f\|_{E_0'} + \hat{\varepsilon}_0, \qquad \|\bar{f}\|_{L_{E_1}'} = \|f\|_{E_1'} + \hat{\varepsilon}_1,$$

$$\|\bar{f}\|_{L_E'} = \|f\|_{E'} + \hat{\varepsilon}, \qquad \text{where } \hat{\varepsilon}_0, \hat{\varepsilon}, \hat{\varepsilon}_1 < \delta.$$

Thus, the relation

$$\|f\|_{E'} + \hat{\varepsilon} \geq (\|f\|_{E_0'} + \hat{\varepsilon}_0)^{1-\alpha_i} (\|f\|_{E_1'} + \hat{\varepsilon}_1)^{\alpha_i}$$

holds for all $\hat{\varepsilon}_0, \hat{\varepsilon}, \hat{\varepsilon}_1 < \delta$, and therefore

$$\|f\|_{E'} \geq \|f\|_{E_0'}^{1-\alpha_i} \|f\|_{E_1'}^{\alpha_i}.$$

The theorem is proved.

Let $E_0 \supset E \supset E_1$ be three normally embedded Banach spaces possessing the properties: 1) the spaces $E_0 \supset E$ and $E \supset E_1$ are related; 2) $\mathrm{ind}_{E_0,E_1}(E) \neq 1$.

It is natural to consider the following problem: will the spaces $E_0 \supset E \supset E_1$ be related if conditions 1) and 2) are satisfied for them? We shall show that, in this general formulation, the indicated problem is answered in the negative. Indeed, it is not difficult to construct two finite-dimensional normally embedded Banach spaces $E_0 \supset E_1$, for which the family of Ba

Banach spaces $\widetilde{E}_\alpha(E_0, E_1)$ with unit balls W_α $(0 \leq \alpha \leq 1)$ does not form a normal scale.

Since the maximal scale $E_\alpha^{\min}(E_0, E_1)$ joining the spaces E_0 and E_1 does not coincide with the scale $\widetilde{E}_\alpha(E_0, E_1)$, there exists a number $\alpha_i \neq 1$ such that

$$\widetilde{E}_{\alpha_i}(E_0, E_1) \subset E_{\alpha_i}^{\max}(E_0, E_1), \tag{5}$$

but

$$\widetilde{E}_{\alpha_i}(E_0, E_1) \neq E_{\alpha_i}^{\max}(E_0, E_1). \tag{6}$$

The spaces $E_0 \supset E_{\alpha_i}(E_0, E_1) \supset E_1$ give an example of three non-kindred normally embedded Banach spaces. Indeed, if one assumes that the spaces $E_0, \widetilde{E}_{\alpha_i}(E_0, E_1), E_1$ are kindred, then

$$\widetilde{E}_{\alpha_i}(E_0, E_1) \supset E_{\alpha_i}^{\max}(E_0, E_1).$$

This latter embedding, together with (5), contradicts condition (6).

Denote by $\mathcal{L}_1(\rho)$ the Banach space of measurable functions defined on the interval $[0, 1]$ and summable with almost everywhere positive weight $\rho(t) \in L_1$.

The space $E_1 = \mathcal{L}_1(\rho_1)$ is normally embedded in $E_0 = \mathcal{L}_1(\rho_0)$ if and only if $\rho_0(t) \leq \rho_1(t)$ for almost all $t \in [0, 1]$.

Theorem 2. *The maximal scale joining the spaces $\mathcal{L}_1(\rho_0) = E_0$ and $\mathcal{L}_1(\rho_1) = E_1$ consists of the spaces $\mathcal{L}_1(\rho_\alpha)$ with weights*

$$\rho_\alpha(t) = [\rho_0(t)]^{1-\alpha}[\rho_1(t)]^\alpha.$$

The index of the intermediate space $\mathcal{L}_1(\rho)$ between the spaces $\mathcal{L}_1(\rho_0)$ and $\mathcal{L}_1(\rho_1)$ is equal to

$$\operatorname{ind}_{\mathcal{L}_1(\rho_0),\, \mathcal{L}_1(\rho_1)}[\mathcal{L}_1(\rho)] = \inf_{\rho(t) \leq [\rho_0(t)]^{1-\alpha}[\rho_1(t)]^{\alpha}} \alpha.$$

Theorem 3. *Let $\mathcal{L}_1(\rho_0) \supset \mathcal{L}_1(\rho) \supset \mathcal{L}_1(\rho_1)$ be Banach spaces of measurable functions defined on the interval $[0,1]$ and integrable with almost everywhere positive summable weights $\rho_0(t), \rho(t), \rho_1(t)$.*

In order that the Banach spaces $\mathcal{L}_1(\rho_0), \mathcal{L}_1(\rho), \mathcal{L}_1(\rho_1)$ be kindred, it is necessary and sufficient that

$$\operatorname{ind}_{\mathcal{L}_1(\rho_0), \mathcal{L}_1(\rho_1)}[\mathcal{L}_1(\rho)] = \alpha_i \neq 1.$$

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