ON THE APPROXIMATION OF A PERIODIC FUNCTION BY A BOUNDED SEMIADDITIVE OPERATOR

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.5

MATHEMATICS

V. V. ZHUK

ON THE APPROXIMATION OF A PERIODIC FUNCTION BY A BOUNDED SEMIADDITIVE OPERATOR

(Presented by Academician V. I. Smirnov on 13 XI 1965)

1°. In this paper the following problems are considered: 1) the determination of structural properties of a function on the basis of its representation by a bounded semiadditive operator; 2) the determination of the order of approximation of a function by a bounded semiadditive operator, if the structural properties are specified by moduli of smoothness.

Among the numerous works devoted to this question, we mention the papers ([1–7]).

2°. Notation and assumptions. The function $f(x) \in C_{2\pi}$; $\omega_k(\delta, f)$ is its modulus of smoothness of order k; $E_n(f)$ is the best approximation by trigonometric polynomials of order $\leq n$; $T_n(x, f)$ is a polynomial of best approximation of order n; $\sigma(f)$ is the Fourier series:

$$\sigma(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx);$$

$S_n(x, f)$ is the partial sum of the Fourier series of order n; $\tilde{f}(x)$ is the function trigonometrically conjugate to $f(x)$; $\tau_n(x, f)$ are the Vallée-Poussin sums

$$\tau_n(x, f) = \frac{S_n(x, f) + \cdots + S_{2n-1}(x, f)}{n};$$

$T_n(x)$ is a trigonometric polynomial of order $\leq n$; $U(f)$ is a bounded semiadditive (subadditive) operator from $C_{2\pi}$ into $C_{2\pi}$; $\omega_r(\delta)$ satisfies the conditions: 1) it is continuous for $0 \leq \delta < \infty$; 2) $\omega_r(\delta_1) \leq \omega_r(\delta_2)$ when $0 \leq \delta_1 \leq \delta_2$; 3) $\omega_r(0) = 0$; 4) $\omega_r(\lambda\delta) \leq (\lambda + 1)^r \omega_r(\delta)$ for any $\lambda \geq 0$; the function $\Phi_\mu(t)$ ($\Phi_n(t)$): a) is summable and bounded on $[-\pi, \pi]$, b) even, c) positive, d)

$$\int_{-\pi}^{\pi} \Phi_\mu(t)\, dt = 1;$$

the numbers r, k, and n are natural; $C(0 < C < \infty)$ and $M(0 \le M < \infty)$ are constants depending only on those arguments which will be indicated,

$$\Delta_{\mu,r} = \int_0^\pi t^r \Phi_\mu(t)\,dt.$$

3°. Main theorems.

Theorem 1. If $\|U(T_n(x,f))\| \le M_1 \|T_n^{(k)}(x,f)\|$, then

$$\|U(f)\| \le (\|U\| + M_1 n^k) E_n(f) + \frac{M_1 n^k}{2^k}\,\omega_k\left(\frac{\pi}{n}, f\right).$$

Theorem 2. If $M_2 \|T_n^{(k)}(x,f)\| \le \|U(T_n(x,f))\|$, then

$$M_2 n^k \omega_k(n^{-1}, f) \le (\|U\| + 2^k M_2 n^k) E_n(f) + \|U(f)\|.$$

Theorem 3. Let $\tilde{f}(x) \in C_{2\pi}$. If

$$\|U(\tau_{[n/2]}(x,f))\| \le M_3 \|\tau_{[n/2]}^{(k)}(x,\tilde{f})\|,$$

then

$$\|U(f)\| \leqslant 4\|U\| E_{[n/2]}(f) + 4M_3 n^{kE}_{[n/2]}(\tilde{f}) + \frac{M_3 n^k}{2^k}\omega_k\left(\frac{\pi}{n}, \tilde{f}\right).$$

Theorem 4. Let $\tilde{f}(x) \in C_{2\pi}$. If

$$M_4 \|T_{[n/2]}^{(k)}(x,\tilde{f})\| \leqslant \|U(\tau_{[n/2]}(x,f))\|,$$

then

$$M_4 n^k \omega_k(n^{-1}, \tilde{f}) \leqslant 4\|U\| E_{[n/2]}(f) + 2^{k+2} n_4^{kM} E_{[n/2]}(f) + \|U(f)\|.$$

Theorem 5. Let $f^{(r)}(x) \in C_{2\pi}$. If for every $T_n(x)$

$$\|U(T_n)\| \leqslant M_5 \|\widetilde{T}_n^{(k)}(x)\|,$$

then

$$\|U(f^{(r)})\| \leqslant C_1(r)\|U\| E_n(f^{(r)}) + C_2(r+k) M_5 \sum_{l=0}^{n-1} (l+1)^{r+k-1} E_l(f).$$

Let $U_k(f)$ $(k = 1, 2)$ be bounded subadditive operators from $C_{2\pi}$ into $C_{2\pi}$, possessing the following properties for every $T_n(x)$:

1) $U_k(T_n)$ is a trigonometric polynomial of order $\leqslant n$;
2) $U_k(T_n) = \widetilde{U}_k(T_n)$;
3) for any r, $U_k^{(r)}(T_n) = U_k(T_n^{(r)})$.

Theorem 6. Let p_k $(k = 1, 2)$ be natural numbers. If, for every $T_n(x)$:

1)

$$M_6(p_k)\|T_n^{(p_k)}(x)\| \leqslant \|U_k(T_n)\| \leqslant M_7(p_k)\|T_n^{(p_k)}(x)\|,$$

then

$$\prod_{k=1}^{2} M_6(p_k)\|T_n^{(p_1+p_2)}(x)\| \leqslant \|U_1(U_2(T_n))\| \leqslant \prod_{k=1}^{2} M_7(p_k)\|T_n^{(p_1+p_2)}(x)\|;$$

2)

$$M_8(p_k)\|\widetilde{T}_n^{(p_k)}(x)\| \leqslant \|U_k(T_n)\| \leqslant M_9(p_k)\|\widetilde{T}_n^{(p_k)}(x)\|,$$

then

$$\prod_{k=1}^{2} M_8(p_k)\|T_n^{(p_1+p_2)}(x)\| \leqslant \|U_1(U_2(T_n))\| \leqslant \prod_{k=1}^{2} M_9(p_k)\|T_n^{(p_1+p_2)}(x)\|;$$

3)

$$M_{10}(p_1)\|T_n^{(p_1)}(x)\| \leqslant \|U_1(T_n)\| \leqslant M_{11}(p_1)\|T_n^{(p_1)}(x)\|,$$

$$M_{10}(p_2)\|\widetilde{T}_n^{(p_2)}(x)\| \leqslant \|U_2(T_n)\| \leqslant M_{11}(p_2)\|\widetilde{T}_n^{(p_2)}(x)\|,$$

then

$$\prod_{k=1}^{2} M_{10}(p_k)\|\widetilde{T}_n^{(p_1+p_2)}(x)\| \leqslant \|U_1(U_2(T_n))\| \leqslant \prod_{k=1}^{2} M_{11}(p_k)\|\widetilde{T}_n^{(p_1+p_2)}(x)\|.$$

We note that Theorem 6 generalizes some results of I. P. Natanson ([8–10]).

Theorem 7. Let $0 < C < 1$, r be an even number,

$$I_{\mu}^{(r)}(x,f)=\frac{(-1)^{r/2+1}2}{C_r^{r/2}}\int_{-\pi}^{\pi}\sum_{k=1}^{r/2}(-1)^{k+r/2}C_r^{k+r/2}f(x+kt)\Phi_{\mu}(t)\,dt.$$

If

$$1\leqslant n\leqslant\left[\sqrt{\frac{24(1-C)\Delta_{\mu,r}}{r\Delta_{\mu,r+2}}}\right],$$

then

$$Cn^r\Delta_{\mu,r}\omega_r(n^{-1},f)\leqslant C_3(r)\big[(1+C\Delta_{\mu,r}n^r)E_n(f)+\|I_{\mu}^{(r)}(x,f)-f(x)\|\big].$$

Theorem 8. Let, for every $T_n(x)$,

$$M_{12}\|\widetilde{T}_n^{(r)}(x)\|\leqslant\|U(T_n)\|.$$

Then

$$M_{12}n^r\omega_{r+1}(n^{-1},f)\leq\left(\|U\|+2^{r+1}M_{12}n^r\right)E_n(f)+\|U(f)\|.$$

Theorem 9. If $f^{(k)}(x)\in C_{2\pi}$, then

$$\omega_r(n^{-1},f^{(k)})\leq C_4(r,k)\left[E_n(f^{(k)})+n^k\omega_{r+k}(n^{-1},f)\right].$$

4°. Some applications of the main theorems to singular integrals. We shall apply the main theorems to the V. A. Steklov integral and to some of its modifications.

Theorem 10. Let r be an even number. If

$$\left\|\frac{1}{\delta}\int_0^{\delta}\sum_{l=0}^{r}(-1)^lC_r^lf\left[x+\left(\frac{r}{2}-l\right)t\right]\,dt\right\|\leq C_5(r)\omega_r(\delta),$$

then

$$\omega_r(\delta,f)\leq C_6(r)\omega_r(\delta).$$

Theorem 11. Let

$$L_1(h, f) = \left\| \frac{1}{2h} \int_{-h}^{h} |f(x+t) - f(x)|\, dt \right\|.$$

Then

$$\omega_1(n^{-1}, f) \le C_7 \left[n^{-3} L_1(\pi, f) + n^{-3} \int_{n^{-1}}^{\pi} h^{-4} L_1(h, f)\, dh \right].$$

Corollary. If $L_1(\delta, f) \le C_8 \omega_1(\delta)$, then $\omega_1(\delta, f) \le C_9 \omega_1(\delta)$.

We now apply the main theorems to the Jackson integral and some of its generalizations.

Theorem 12. Let r be an even number,

$$D_n^{(r)}(f) = \frac{(-1)^{r/2+1} 2}{C_r^{r/2}} \int_{-\pi}^{\pi} \sum_{k=1}^{r/2} (-1)^{k+r/2} C_r^{k+r/2} f(x+kt) \Phi_n^{(r)}(t)\, dt,$$

where

$$\Phi_n^{(r)}(t) = \left[\int_{-\pi}^{\pi} \left(\frac{\sin nt/2}{\sin t/2} \right)^{2r+2} dt \right]^{-1} \left(\frac{\sin nt/2}{\sin t/2} \right)^{2r+2}.$$

Then

$$\omega_r(n^{-1}, f) \le C_{10}(r) \sup_{m \ge n} \| f(x) - D_m^{(r)}(f) \| \le C_{11}(r) \omega_r(n^{-1}, f).$$

Theorem 13. Suppose that for $0 \le t \le n^{-1}$ the kernel $\Phi_n|t| \ge C_{12} n$,

$$\Phi_n(t) = \frac{1}{2\pi} + \sum_{k=1}^{n} \rho_k(n) \cos kt.$$

Then

$$\omega_2(n^{-1}, f) \le C_{13} \left\| \int_0^{\pi} |f(x+t) - 2f(x) + f(x-t)| \Phi_n(t)\, dt \right\|,$$

$$\omega_1(n^{-1}, f) \le C_{14} \left\| \int_{-\pi}^{\pi} |f(x+t) - f(x)| \Phi_n(t)\, dt \right\|.$$

Remark. For the Jackson kernel the conditions of the theorem are satisfied.

Theorem 14. Let $\Phi_n(t)$ be a trigonometric polynomial of order $\leq n$,

$$C_{15} n^{-2} \leq \Delta_{n,2} \leq C_{16} n^{-2}, \qquad C_{17} n^{-4} \leq \Delta_{n,4} \leq C_{18} n^{-4}.$$

Then

$$\omega_2(n^{-1}, f) \leq C_{19} \sup_{m \geq n} \left\| \int_{-\pi}^{\pi} \{f(x+t) - f(x)\} \Phi_m(t)\, dt \right\|.$$

All the results given are also valid for the space $L^p_{2\pi}$ $(1 \leq p < \infty)$.

5°. Approximation in the space $L^p_{2\pi}$ $(1 < p < \infty)$.

We shall adopt the following assumptions. The function* $f(x) \in L^p_{2\pi}$, where $1 < p < \infty$; the numbers $\gamma_k(n)$ are such that

$$|\gamma_{k+1}(n)| \leq C_{20}, \qquad \sum_{l=2^k}^{2^{k+1}-1} |\gamma_l(n) - \gamma_{l+1}(n)| \leq C_{21} \quad (k = 0, 1, ...).$$

Put

$$U_n(x, f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \gamma_k(n)(a_k \cos kx + b_k \sin kx).$$

Theorem 15. Let

$$\rho_k(n) = \frac{[1 - \gamma_k(n)]\, n^r}{k^m} \qquad \text{for } 1 \leq k \leq [n^{r/m}],$$

$$\rho_k(n) = 0 \qquad \text{for } k > [n^{r/m}].$$

If

$$|\rho_{k+1}(n)| \leq C_{22}, \qquad \sum_{l=2^k}^{2^{k+1}-1} |\rho_{l+1}(n) - \rho_l(n)| \leq C_{23} \quad (k = 0, 1, ...),$$

then

$$\|U_n(x, f) - f(x)\|_{L^p} \leq C_{24}(p)\, \omega_m([n^{r/m}]^{-1}, f)_{L^p}.$$

Theorem 16. Let $0 \leq \gamma_{k+1}(n) < 1$ $(k = 0, 1, ...)$,

$$\rho_k(n) = \frac{k^m}{[1-\gamma_k(n)]\,n^r} \qquad \text{for } 1 \le k \le [n^{r/m}],$$

$$\rho_k(n) = 0 \qquad \text{for } k > [n^{r/m}].$$

If

$$|\rho_{k+1}(n)| \le C_{25}, \qquad \sum_{l=2^k}^{2^{k+1}-1} |\rho_{l+1}(n) - \rho_l(n)| \le C_{26} \quad (k = 0, 1, ...),$$

then

$$\omega_m([n^{r/m}]^{-1}, f)_{L^p} \le C_{27}(p)\big[E_{[n^{r/m}]}(f) + \|U_n(x,f) - f(x)\|_{L^p}\big].$$

As an example of applications of Theorems 15 and 16, we give one theorem.

Theorem 17. Let $0 \le u < 1$,

$$P_u(x,f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} u^k (a_k \cos kx + b_k \sin kx)$$

be the Poisson operator. Then

$$\omega_1(1-u, f)_{L^p} \le C_{28}(p)\|P_u(x,f) - f(x)\|_{L^p} \le C_{29}(p)\omega_1(1-u,f)_{L^p}.$$

The author expresses deep gratitude to G. I. Natanson for his attention to this work.

Leningrad State University
named after A. A. Zhdanov

Received
8 XI 1965

References

1. S. M. Lozinskii, DAN, 85, 5, 645 (1952).
2. S. B. Stechkin, Izv. AN SSSR, ser. matem., 15, No. 3, 219 (1951).
3. S. B. Stechkin, Tr. Matem. inst. im. V. A. Steklova AN SSSR, 62, 48 (1961).

4. M. F. Timan, Izv. AN SSSR, ser. matem., 29, No. 3, 587 (1965).

5. A. Kh. Turetskii, Izv. AN SSSR, ser. matem., 25, No. 3, 411 (1961).

6. R. M. Trigub, Izv. AN SSSR, ser. matem., 29, No. 3, 615 (1965).

7. F. I. Kharshiladze, DAN, 122, No. 3, 352 (1958).

8. I. P. Natanson, DAN, 82, No. 3, 337 (1952).

9. I. P. Natanson, DAN, 158, No. 3, 523 (1964).

10. I. P. Natanson, Izv. vyssh. uchebn. zaved., ser. Matematika, No. 3, 93 (1964).

* In this section $f(x)$ is not assumed to belong to $C_{2\pi}$.

Note: Figure translations are in progress. See original paper for figures.