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Abstract

Full Text

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On the Question of Defining the Concept of a Solution of an Integral Equation with a Discontinuous Operator

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1. Consider the equation

$$x(t) = \lambda \int_0^1 K[t, s, x(s)]\, J[s, x(s)]\, ds \tag{1}$$

under the following assumptions.

$K(t, s, x)$ is defined in $G : 0 \leq t, s \leq 1,\ |x| < c$, is measurable in s for all t and x, and is continuous in x for all t and almost all s. For any positive $\gamma < c$, there exists a function $\mu_\gamma(t, s)$, summable in s and continuous in t on $[0, 1]$, such that $|K(t, s, x)| \leq \mu_\gamma(t, s)$ for $|x| \leq \gamma$. The function $J(s, x) \in L_\infty$ in the closed domain $G_\gamma : 0 \leq s \leq 1,\ |x| \leq \gamma$, for every $\gamma < c$.

A definition of a solution of equation (1) was proposed in ([1,2]). However, in the discussion of the aforementioned works at the Izhevsk seminar it became clear that the definition of A. B. Samarov and Lee Mun Su has a number of essential shortcomings. For example, when a differential equation $y' = J(t, y)$ is reduced by Fubini' s method to an integral equation, solutions arise that do not satisfy, in the sense of works ([3,4]), the original differential equation; in the case of a variable upper limit, the theorem on continuation of a solution is false.

To formulate the new definition proposed by us, denote

$$M_x\{J(s, x)\} = \lim_{\delta \to 0} \inf_{\mu N = 0} \sup_{z \in R(x,\delta) - N} J(s, z),$$

$$m_x\{J(s, x)\} = \lim_{\delta \to 0} \sup_{\mu N = 0} \inf_{z \in R(x,\delta) - N} J(s, z),$$

where $R(x, \delta)$ is the interval of radius δ centered at the point x.

We shall call $v(t) \in C_{[0,1]}$ a solution of equation (1) if

$$|v(t)| < c, \qquad v(t) = \lambda \int_0^1 K[t, s, v(s)]\, R_v(s)\, ds,$$

where, for almost all $s \in [0, 1]$,

$$m_x\{J(s,v(s))\} \le R_v(s) \le M_x\{J(s,v(s))\}. \tag{2}$$

With this definition, the following assertions on existence and estimates of solutions are valid.

Theorem 1. *Equation (1) has a solution if*

$$0 < \lambda < \gamma \Bigg/ \left\| \int_0^1 \mu_\gamma(t,s) \ \underset{x\in G_\gamma(s)}{\operatorname{vrai\,max}} |J(s,x)|\, ds \right\|_{C[0,1]}.$$

Theorem 2. *Let $K(t,s,x) \ge 0$ and be nondecreasing in x in G. Suppose, furthermore,*

$$m_x\{J(s,x(s))\} \ge A_1(s,x(s)) - A_2(s,x(s)),$$

$$M_x\{J(s,x(s))\} \le A_3(s,x(s)) - A_4(s,x(s)),$$

where $A_i \in L_\infty(G)$ $(i = 1,2,3,4)$ and are nondecreasing in x. If a pair of functions $z_i(t)$, continuous on $[0,1]$, $|z_i(t)| < c$, $i = 1,2$, $z_2(t) > z_1(t)$, satisfies the integral inequalities

$$z_1(t) < \lambda \int_0^1 K[t,s,z_1(s)]\, m_x\{A_1(s,z_1(s))\}\, ds -$$

$$-\lambda \int_0^1 K[t,s,z_2(s)]\, M_x\{A_2(s,z_2(s))\}\, ds,$$

(3)

$$z_2(t) > \lambda \int_0^1 K[t,s,z_2(s)]\, M_x\{A_3(s,z_2(s))\}\, ds -$$

$$-\lambda \int_0^1 K[t,s,z_1(s)]\, m_x\{A_4(s,z_1(s))\}\, ds,$$

then there exists a solution $u(t)$ of equation (1), with $z_1(t) \le u(t) \le z_2(t)$.

Remark 1. If $J = J_1 - J_2$ and in G the functions J_i and KJ_i $(i = 1,2)$ are nondecreasing in x, then condition (3) may be replaced by the inequalities

$$z_1(t) < \lambda \int_0^1 K[t, s, z_1(s)]\, m_x\{J_1(s, z_1(s))\}\, ds -$$

$$-\lambda \int_0^1 K[t, s, z_2(s)]\, M_x\{J_2(s, z_2(s))\}\, ds,$$

$$z_2(t) > \lambda \int_0^1 K(t, s, z_2(s)]\, M_x\{J_1(s, z_2(s))\}\, ds -$$

$$-\lambda \int_0^1 K[t, s, z_1(s)]\, m_x\{J_2(s, z_1(s))\}\, ds.$$

Remark 2. The last inequalities are equivalent to the condition

$$\lambda \int_0^1 K[t, s, z_2(s)]\, M_x\{J_2(s, z_2(s))\}\, ds -$$

$$-\lambda \int_0^1 K[t, s, z_1(s)]\, m_x\{J_2(s, z_1(s))\}\, ds < \min(-\varphi_1, \varphi_2),$$

where

$$\varphi_i = z_1 - \lambda \int_0^1 K[t, s, z_i(s)]\, m_x\{J_1(s, z_i(s))\}\, ds +$$

$$+\lambda \int_0^1 K[t, s, z_i(s)]\, m_x\{J_2(s, z_i(s))\}\, ds, \qquad i = 1, 2.$$

Remark 3. For $J \equiv 1$ and under the condition that $K(t, s, x)$ decreases in x, from Theorem 2 and the preceding remarks one obtains, as a special case, a number of assertions of works (5–8), which underlie certain new estimates of solutions of differential equations.

2. For the Volterra equation

$$x(t) = \int_0^t K[t, s, x(s)] J[s, x(s)]\, ds \tag{4}$$

we additionally assume that $|K(t, s, x) - K(t_1, s, x)| \leq v_\gamma(t, t_1, s)$, and for any $\varepsilon > 0$ one can find such a $\delta > 0$ that

$$\int_0^t v_\gamma(t,t_1,s)\,ds + \int_t^{t_1} \mu_\gamma(t_1,s)\,ds < \varepsilon \quad \text{for } |t-t_1|<\delta,\ |x|\le\gamma.$$

The local existence theorem and the theorem on continuation of the solution of equation (4) are valid. Under the condition of monotone decrease with respect to x of the functions $J(s,x)$ and $K(t,s,x)J(s,x)$, the theorems of paper (5) on the existence of upper and lower solutions and the theorem on the integral inequality carry over.

3. For an equation of the form

$$x(t) = \int_0^t K(t,s)J(s,x(s))\,ds, \tag{5}$$

arising when reducing a differential equation to an integral one, inequality (3) is equivalent to the condition

$$R(s) \in \prod_{\delta>0}\prod_{\mu N=0} \operatorname{konv} J\{s, V(u(s),\delta) - N\}$$

for almost all $s \in [0,t]$.

Lemma. In order that the function $u(t)$ be a solution of equation (5) on $[0,1]$, it is necessary and sufficient that, for any prescribed $\varepsilon > 0$ and $\bar{N} \subset G$ ($\mu N = 0$), there exist a function $\psi(t)$, measurable on $[0,1]$, such that $B(t,\psi(t)) \subset G$; $J(t,\psi(t))$ is summable on $[0,1]$,

$$|u(t)-\psi(t_0)| < \varepsilon, \quad \left|\psi(t) - \int_0^t K(t,s)J(s,\psi(s))\,ds\right| < \varepsilon \quad \text{on } [0,1],$$

and $(t,\psi(t)) \in \overline{N}$ for almost all $t \in [0,1]$.

For equation (5), the existence of upper and lower solutions is guaranteed without the monotonicity conditions stipulated in item 2; namely, the following is valid.

Theorem 3. Let $K(t,s) \ge 0$ for $0 \le s \le t \le h$, and let h be sufficiently small. Then (5) has, on $[0,h]$, an upper solution $u_1(t)$ and a lower solution $u_2(t)$, i.e. $u_1(t) \ge u(t) \ge u_2(t)$ on $[0,h]$ for any solution $u(t)$.

4. Consider the Cauchy problem

$$\mathcal{L}[y] = J(t,y), \qquad y^{(k)}(0) = 0, \quad k = 0,\dots,n-1, \tag{6}$$

where $J(t,y)$ is defined above; $\mathcal{L}(y) = y^{(n)} - \sum_{k=0}^{n-1} g_k(t)y^{(k)}$; $g_k(t)$ are functions summable on $[0,1]$. Let $p(t)$ be summable on $[0,1]$, and let $K(t,s)$ be the Cauchy function ([7,9]) of the equation $\mathcal{L}(y) = p(t)y$. Writing (6) in the form

$$\mathcal{L}[y] - p(t)y = J(t,y) - p(t)y, \qquad y^{(k)}(0) = 0,\ k = 0,\dots,n-1,$$

we obtain the equation equivalent to (6)

$$y(t) = \int_0^t K(t,s)\{J(s,y(s)) - p(s)y(s)\}\,ds. \tag{7}$$

It is known that for $n = 1$ problem (6) has upper and lower solutions. In paper ([9]) the existence of upper and lower solutions for any n was proved under the assumption that $J(t,y)$ is discontinuous and satisfies condition L_1 ([1,9,10]). Directly from (7) and Theorem 3 it follows that problem (6) has upper and lower solutions. Hence it is clear that the existence of such solutions of problem (6) depends not on the order n and not on condition L_1, but on

of the form of the right-hand side of the equation (the right-hand side contains no derivatives of y).

Let us note the following consequence of Theorem 3.

Corollary. Suppose that $J(t,y)$ satisfies condition L_2 ([10,11]), i.e. $J(t,y) = p(t)y - H(t,y)$, where $H(t,y)$ is nonincreasing in y, and $p(t)$ is summable. Then problem (6) has at most one solution.

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REFERENCES

[1] Li Mun Su, *Differential Equations*, **1**, No. 8 (1965).
[2] Li Mun Su, A. B. Samarov, *Abstracts of the Graduate-Student Conference of Kazan University*, issue 4, 1964.
[3] E. E. Viktorovskii, *Mat. Sb.*, **34** (76), 213 (1954).
[4] A. F. Filippov, *Mat. Sb.*, **51** (93), No. 1, 99 (1960).
[5] N. V. Azbelev, Z. B. Tsalyuk, *Mat. Sb.*, **56** (98), No. 3, 325 (1962).
[6] Yu. V. Komlenko, *Reports of the Third Siberian Conference on Mathematics and Mechanics*, Tomsk, 1964.
[7] Yu. V. Komlenko, *Dokl. Akad. Nauk SSSR*, **164**, No. 2 (1965).
[8] N. V. Azbelev, *Scientific Reports of Higher School, Physico-Mathematical Sciences*, No. 6 (1958).
[9] N. V. Azbelev, Z. B. Tsalyuk, *Differential Equations*, **1**, No. 4 (1965).

[10] G. P. Kukhta, *Scientific Notes of Kishinev University*, **29** (1957).
[11] S. A. Pak, *Siberian Mathematical Journal*, **3**, No. 4 (1962).

Note: Figure translations are in progress. See original paper for figures.