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Abstract

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MATHEMATICAL PHYSICS

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A METHOD FOR SOLVING THE GENERAL BOUNDARY-VALUE PROBLEM OF THE PROPAGATION OF LONG AND SUPER-LONG RADIO WAVES AROUND THE EARTH

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1. The basic features of the waves under consideration are contained in the following boundary-value problem. In an unbounded medium, described by Maxwell's equations in spherical coordinates r, θ, φ with dielectric-constant tensor $\varepsilon((\theta, \varphi, r))$ and divided into 3 regions: 1) the Earth, $0 < r < a(\theta, \varphi)$, where ε degenerates into a scalar $\varepsilon(\theta, \varphi, r) = \varepsilon' + i\varepsilon''$, $\varepsilon'' \neq 0$; 2) the atmosphere, $a(\theta, \varphi) < r < c(\theta, \varphi)$, where $\varepsilon \equiv 1$, and 3) the ionosphere, $c(\theta, \varphi) < r < \infty$, where ε is arbitrary, but as $r \rightarrow \infty$, $\varepsilon \rightarrow \varepsilon = \text{const}$, one seeks the amplitudes $\mathbf{E}(M)$ and $\mathbf{H}(M)$ of the electromagnetic fields produced by a Hertz dipole $P\delta(\theta, r - b)\exp(-i\omega t)$ (δ is the delta function). For simplicity of exposition we assume ε independent of φ and equal to $\varepsilon(\theta, \varphi = \Phi, r)$, where $\varphi = \bar{\varphi} = \text{const}$ is the plane passing through the polar axis and the observation point M . Then, introducing potentials $A(\theta, r)$ and $B(\theta, r)$, related to $H_\varphi = \partial B/r\partial\theta$ and $E_\varphi = \partial A/r\partial\theta$, and their derivatives with respect to θ , $C = \partial B/\partial\theta$ and $D = \partial A/\partial\theta$, we write the boundary-value problem in the form

$$\frac{\partial}{\partial\theta} \cdot \mathbf{B} - L_{r\theta} \cdot \mathbf{B} = P\delta(\theta, r - b), \quad (1)$$

where $\mathbf{B}(\theta, r)$ is a one-column matrix with elements B, A, C, D , through which all components of the amplitudes of the fields \mathbf{E} and \mathbf{H} are expressed, called below a vector-function; $L_{r\theta}$ is a differential operator represented by the 4×4 matrix $\|l_{ji}\|$, where

$$l_{11} = l_{12} = l_{14} = l_{21} = l_{22} = l_{23} = 0; \quad l_{13} = l_{24} = 1;$$

$$l_{31} = -\frac{r^2\Delta}{\varepsilon_{\theta\theta}} \left[\frac{\partial}{\partial r} \left(\frac{\varepsilon_{rr}}{\Delta} \frac{\partial}{\partial r} \right) + k^2 \right]; \quad l_{32} = -\frac{ikr^2\Delta}{\varepsilon_{\theta\theta}} \frac{\partial}{\partial r} \left[\frac{\varepsilon^*}{\Delta} \right];$$

$$l_{33} = -\frac{r^2 \Delta}{\varepsilon_{\theta\theta}} \left[\frac{\partial}{\partial r} \left(\frac{\varepsilon_{\theta r}}{r \Delta} \cdot \right) + \frac{\varepsilon_{r\theta}}{r \Delta} \frac{\partial}{\partial r} \cdot + \text{ctg } \theta \cdot \right]; \quad l_{34} = \frac{ikr \varepsilon^{**}}{\varepsilon_{\theta\theta}}; \quad (2)$$

$$l_{41} = \frac{ikr^2 \bar{\varepsilon}^*}{\Delta} \frac{\partial}{\partial r} \cdot; \quad l_{42} = -r^2 \left[\frac{\partial^2}{\partial r^2} \cdot + k^2 \tilde{\varepsilon} \cdot \right];$$

$$l_{43} = \frac{ikr \bar{\varepsilon}^{**}}{\Delta} \cdot; \quad l_{44} = -\text{ctg } \theta \cdot;$$

$$\Delta = \varepsilon_{\theta\theta} \varepsilon_{rr} - \varepsilon_{\theta r} \varepsilon_{r\theta}; \quad \tilde{\varepsilon} = \varepsilon_{\varphi\varphi} + (\varepsilon_{\varphi\theta} \varepsilon^* + \varepsilon_{\varphi r} \varepsilon^{**}) / \Delta;$$

$$\varepsilon^* = \varepsilon_{r\varphi} \varepsilon_{\theta r} - \varepsilon_{\theta\varphi} \varepsilon_{rr}; \quad \varepsilon^{**} = \varepsilon_{r\theta} \varepsilon_{\varphi\theta} - \varepsilon_{\theta\theta} \varepsilon_{r\varphi}; \quad (3)$$

$$\bar{\varepsilon}^* = -\varepsilon_{\varphi r} \varepsilon_{r\theta} + \varepsilon_{\varphi\theta} \varepsilon_{rr}; \quad \bar{\varepsilon}^{**} = \varepsilon_{\theta r} \varepsilon_{\varphi\theta} - \varepsilon_{\theta\theta} \varepsilon_{\varphi r},$$

under the conditions of continuity of \mathbf{B} on the interfaces of the regions $r = a$ and $r = c$, and boundedness as $r \rightarrow 0$, $r \rightarrow \infty$, $\theta \rightarrow 0$, $\theta \rightarrow \pi$.

Equation (1) is approximate. In it an integral has been omitted, under whose sign stand products of the elements of \mathbf{B} by derivatives of the components of ε with respect to θ .

§ 2. **Method of coupled lines.** We seek the solution of (1) in the form

$$\mathbf{B} = \sum \mathbf{Y}_k(r) \cdot \mathbf{b}_k(\theta), \quad (4)$$

where \mathbf{b}_k is a vector function with elements b_k, a_k, c_k, d_k , and \mathbf{Y}_k is a vector function with elements Y_k, Z_k, U_k, V_k , which is an eigenfunction of the operator $L_{r\bar{\theta}}$, obtained from $L_{r\theta}$ by “freezing” the coefficients in $\|v_{ji}\|$ with respect to θ ; θ enters into $L_{r\bar{\theta}}$ as a parameter. The \mathbf{Y}_k are determined from the equation

$$L_{r\bar{\theta}} \cdot \mathbf{Y} = \lambda \mathbf{Y} \quad \text{under the conditions} \quad (\mathbf{Y}_k, \mathbf{Y}_r^*) = \delta_{kr}; \quad (5)$$

$\lambda(\theta)$ are the eigenvalues of $L_{r\bar{\theta}}$; they lie in the 2nd and 4th quadrants of the λ -plane. We number them in order of increasing modulus, respectively $k = 1, 2, \dots$ and $k = -1, -2, \dots$. Parentheses in (5) denote scalar products, and the asterisk denotes eigenfunctions of the adjoint operator. Introduce $L_{r\bar{\theta}}$ into (1):

$$\frac{\partial}{\partial \theta} \cdot \mathbf{B} - L_{r\bar{\theta}} \cdot \mathbf{B} + [L_{r\bar{\theta}} - L_{r\theta}] \cdot \mathbf{B} = \mathbf{P} \delta(\theta, r - b). \quad (6)$$

Substituting (4) into (6), multiplying scalarly by \mathbf{Y}_r^* , and taking (5) into account, we obtain a system of coupled-line equations (1), integrable on a computer:

$$\frac{d}{d\theta} \cdot \mathbf{b}_k - i\nu_k(\theta)\mathbf{b}_k + \sum_{r=-\infty}^{\infty} \|S_{ji}^{(k,r)}(\theta)\| \cdot \mathbf{b}_r = P_k \delta(\theta); \quad \lambda_k = i\nu_k. \quad (7)$$

S_{ji} are 4×4 matrices of complex numbers formed from the scalar products of $[L_{r\bar{\theta}} - L_{r\theta}] \cdot \mathbf{Y}_k$ with \mathbf{Y}_r^* . If the spectrum of $L_{r\bar{\theta}}$ has a continuous part, then an integral over ν also enters into (7). For small S_{ji} and $\nu_k \neq \nu_r$, i.e., far from the cones of spatial resonance⁽¹²⁾, neglecting S_{ji} , we obtain the solution in the form of a sum of modulated normal waves*

$$\mathbf{B} \exp(-i\omega t) = \sum_{k=-\infty}^{+\infty} C_k \mathbf{Y}_k(r, \theta) \exp \left[-i \left(\omega t - \int \nu_k d\theta \right) \right]. \quad (4')$$

The S_{ji} in (7) take into account the interaction of normal waves on sections of the path where ε depends on θ , for example in the sunrise and sunset belt; and also because of the inhomogeneity of the metric of space in spherical coordinates. It should be noted that, in a nonvertical magnetic field of the Earth \mathbf{H}_0 , when $\varepsilon_{\theta r}$ and $\varepsilon_{r\varphi}$ are not equal to 0, $\nu_k \neq -\nu_{-k}$, and therefore the reciprocity principle in wave propagation with respect to θ is violated (see⁽¹¹⁾).

§ 3. To find the wave numbers ν_k , it is convenient to eliminate C and D from (1) and pass to the coupled-line equations containing second-order derivatives with respect to θ . Then (5) becomes the following boundary-value problem for eigenvalues of the parameter ν :

$$\begin{aligned} Y''_{rr} + aY'_r + bY + cZ'_r + dZ &= 0; \\ Z''_{rr} + eZ + fY'_r + gY &= 0, \end{aligned} \quad (5')$$

where a prime denotes differentiation with respect to r ;

$$\begin{aligned} a &= \frac{\Delta}{\varepsilon_{rr}} \left[\left(\frac{\varepsilon_{rr}}{\Delta} \right)'_r \pm ik \frac{(\varepsilon_{\theta r} + \varepsilon_{r\theta})}{\Delta} S \right] \\ b &= \frac{k^2 \Delta}{\varepsilon_{rr}} \left[1 - \frac{\varepsilon_{\theta\theta}}{\Delta} S^2 \pm \frac{ir}{k} \left(\frac{\varepsilon_{\theta\theta}}{r\Delta} \right)'_r S \right]; \\ c &= \frac{ik\varepsilon^*}{\varepsilon_{rr}}; \quad d = ik \left(\frac{\varepsilon^*}{\Delta} \right)'_r \frac{\Delta}{\varepsilon_{rr}} \pm \frac{k^2 \varepsilon^{**}}{\varepsilon_{rr}} S; \quad e = k^2 (\tilde{\varepsilon} - S^2); \end{aligned}$$

* Normal waves ^(1-4,11) are also called free, proper, and, in English, residue waves and modes ⁽⁵⁻⁷⁾, which is not entirely apt and is sometimes translated as “modes.”

$$f = -\frac{ik\varepsilon^*}{\Delta}; \quad g = \pm \frac{\bar{k}^2 \varepsilon^{**}}{\Delta} S; \quad S = \frac{\nu}{kr}.$$

It is necessary to find ν that ensures a nonzero solution of (5') under the boundedness conditions $|Y|$ and $|Z|$ as $r \rightarrow 0$ and $r \rightarrow \infty$. On an electronic computer it is convenient to solve (5') by the method of sweeping the conditions at $r \rightarrow 0$ and $r \rightarrow \infty$ to some point $r = \bar{r}$, where Y, Z, Y'_r, Z'_r must be continuous. Such a method is equivalent to the more economical method of sweeping and matching the impedances $|z|$ of the surface $\bar{r} = \text{const}$ for the regions (\bar{r}, ∞) and $(\bar{r}, 0)$, denoted by $|z(\bar{r} + 0)|$ and $|z(\bar{r} - 0)|$, and determined by the expressions

$$E_\theta = z'_{11}(\bar{r} \pm 0)H_\theta + z'_{12}(\bar{r} \pm 0)H_\varphi; \quad E_\varphi = z'_{21}(\bar{r} \pm 0)H_\theta + z'_{22}(\bar{r} \pm 0)H_\varphi;$$

$|z|$ are expressed in terms of Y, Y'_r, Z , and Z'_r . The matching equation for $|z|$ will be:

$$\det \begin{vmatrix} z'_{11}(\bar{r} + 0) - z'_{11}(\bar{r} - 0) & z'_{12}(\bar{r} + 0) - z'_{12}(\bar{r} - 0) \\ z'_{21}(\bar{r} + 0) - z'_{21}(\bar{r} - 0) & z'_{22}(\bar{r} + 0) - z'_{22}(\bar{r} - 0) \end{vmatrix} = 0. \quad (8)$$

Its roots will be ν_k . We choose as \bar{r} the surface $\bar{r} = c$. To find $|z|$ from (5'), introduce the impedance functions u, χ : $Y = \exp \int u dr$, $Z = \chi Y$; substituting them into (5'), we obtain

$$u'_r + u^2 + au + b + c(\chi'_r + u\chi) + d\chi = 0,$$

$$\chi''_{rr} + 2u\chi'_r + u^2\chi + u'_r\chi + e\chi + fu + g = 0. \quad (9)$$

To compute $|z^\nu(c + 0)|$, we integrate (9) on the computer from r_∞ to $r = c$, where $r_\infty \gg c$ is chosen in the region in which ε is practically constant. The initial values $u(r_\infty)$ are obtained from equation (9) for $u'_r = \chi'_r = \chi''_{rr} = 0$:

$$u^4 + \bar{a}u^3 + (\bar{e} + \bar{b} - \bar{c}\bar{f})u^2 + (\bar{e}\bar{a} - \bar{c}\bar{g} - \bar{d}\bar{f})u + (\bar{e}\bar{b} - \bar{d}\bar{g}) = 0, \quad (10)$$

which passes into Booker's equation [8] as $r \rightarrow \infty$, if the relation of ε to the plasma parameters is determined from the Lorentz equation (formula (1) from [11]). Of the four roots of (10) we choose $u^o(r_\infty)$ and $u^e(r_\infty)$, corresponding to ordinary and extraordinary waves decaying at $+\infty$. The initial $\chi(r_\infty)$ for these u are found from the formula $\chi = -(\bar{g} + \bar{f}u)/(\bar{e} + u^2)$. Integration of (9) from

$r = r_\infty$ in the direction of decreasing r , i.e., toward the traveling waves, signifies a continuous transformation of the impedances of waves of types o and e

$$Z_y^{e,o}(r) = \frac{E_\theta}{H_\varphi} = \frac{1}{\Delta} \left[\frac{\varepsilon_{rr}}{ik} u^{e,o}(r) + \varepsilon^* \chi^{e,o}(r) \pm \varepsilon_{\theta r} S \right];$$

$$Z_z^{e,o}(r) = \frac{H_\theta}{E_\varphi} = -\frac{1}{ik} \left[u^{e,o}(r) + \frac{\chi_{r'}^{e,o}}{\chi^{e,o}} \right]; \quad X^{e,o}(r) = \frac{E_\varphi}{H_\varphi} = \chi^{e,o}(r) \quad (11)$$

from the adiabatic values at $r = r_\infty$ to the values on the surface $r = \bar{r}$. These impedances are related to the previously introduced impedances z_{jk} of the surface $r = \bar{r}$ by the expressions

$$z_{11} = (Z_y^e - Z_y^o)/\delta; \quad z_{12} = (X^e Z_z^e Z_y^o - X^o Z_z^o Z_y^e)/\delta;$$

$$z_{21} = (X^e - X^o)/\delta, \quad z_{22} = (X^e - X^o) Z_z^e Z_z^o / \delta, \quad (12)$$

where $\delta = X^e Z_z^e - X^o Z_z^o$. Carrying the integration to the point $r = c$, we obtain, from formulas (11), (12), the impedances $z_{jk}(c+0)$. The impedances $z_{jk}(c-0)$ are found in two stages. First, for two values $u^{o,e}(r_0)$, $\chi^{o,e}(r_0)$, where $r_0 \ll a$, corresponding to waves traveling toward the center of the Earth, one integrates (9) from $r = r_0$ to the Earth's surface $r = a$ and finds, analogously to the case of the ionosphere, $z_{12}(a-0) = Z_y^e = Z_y^o$ and $z_{21}^{-1}(a-0) = Z_z^e = Z_z^o$. Because of the isotropy of the Earth, $z_{11}(a-0) = z_{22}(a-0) = 0$. The transformation of impedances over the next interval (a, c) is carried out analytically using the formulas

$$iz_{12}^\nu(c-0) = \frac{D_\nu(a', c') - iz_{12}^\nu(a-0)D_\nu(a, c')}{D_\nu(a', c) - iz_{12}^\nu(a-0)D_\nu(a, c)} = \frac{\overline{D_\nu(a', c')}}{D_\nu(a', c)}, \quad (13)$$

$$iz_{21}^\nu(c-0) = \frac{D_\nu(a, c) - iz_{21}^\nu(a-0)D_\nu(a', c)}{D_\nu(a, c') - iz_{21}^\nu(a-0)D_\nu(a', c')} = \frac{\overline{D_\nu(a, c)}}{D_\nu(a, c')},$$

where $D_\nu(a, c)$ are two-argument functions formed from products of the modified Hankel functions $h_\nu^{(1,2)}(ka)$ and $h_\nu^{(1,2)}(kc)$ ⁽²⁻⁴⁾; $z_{11}^\nu(c-0)$ and $z_{22}^\nu(c-0)$ are equal to zero. Substituting (11), (12), (13) into (8), we obtain the final equation for determining the wave numbers of the normal waves

$$\left[\overline{D_\nu(a', c')} - iZ_y^e \overline{D_\nu(a', c)} \right] \left[D_\nu(a, c') + iZ_z^o D_\nu(a, c) \right] -$$

$$-(X^e/X^o)[D_\nu(a', c') - iZ_y^o D_\nu(a', c)] [\overline{D_\nu(a, c')} + iZ_z^e \overline{D_\nu(a, c)}] = 0. \quad (14)$$

Equation (14), together with (9) and (11), was used in ^(2,3) for the case of vertical \mathbf{H}_0 , when (10) becomes biquadratic. The values of \bar{u} and $\bar{\chi}$ for this case are given by formulas (2.40)–(2.41) in ⁽³⁾. If the ionosphere is homogeneous, i.e., $N_e = 0$ for $r < c$ and $N_e = \bar{N}_e$ for $r > c$, then there is no need to integrate (9), since the impedances are obtained directly from (11):

$$Z_y^{e,o}(c) = \frac{1}{\Delta} \left[\frac{\varepsilon_{rr}}{ik} \bar{u}^{e,o} + \varepsilon^* \bar{\chi}^{e,o} \mp \varepsilon_{\theta r} S \right]; \quad Z_z^{e,o} = -\frac{1}{ik} \bar{u}^{e,o}; \quad X^{e,o} = \bar{\chi}^{e,o}. \quad (15)$$

Equation (14) with these impedances was used in work ⁽⁴⁾ for the case of the vertical magnetic field of the Earth, when $\varepsilon_{\theta r} = 0$, $\varepsilon^* = -\varepsilon_{\theta\varphi} \varepsilon_{rr}$, and $\Delta = \varepsilon_{\theta\theta} \varepsilon_{rr}$. The values of $\bar{u}^{e,o}$ and $\bar{\chi}^{e,o}$ in this case are determined from the biquadratic equation (10) by formulas (2.40)–(2.41) of work ⁽³⁾. In works ⁽²⁻⁴⁾, in a neighborhood of $\nu = ka$ and $\nu = kc$, the functions $h_\nu^{(1)}$ and $h_\nu^{(2)}$ in (14) were approximated by Airy functions:

$$h_\nu^{(1,2)}(z) = z^{1/6} h_{1,2}(\zeta); \quad \zeta = -\eta(\nu - z); \quad \eta = \sqrt[3]{z/2} \quad (16)$$

by the “comparison-equation” method ⁽⁹⁾. In the first of the works ⁽⁴⁾, where hand calculation was used, we used tables of the functions $h_{1,2}(\zeta)$ ⁽¹⁰⁾ with 8 significant figures for the complex argument ζ in the circle $|\zeta| \leq 6$. In ^(2,3), for calculating $h_{1,2}(\zeta)$, series given in the foreword to ⁽¹⁰⁾ were used.

Introducing ν_k and Y_k into (4'), we obtain the first approximation to the solution of the boundary-value problem. The solution (7), with allowance for the terms S_{ji} , will give the second approximation. To refine the interaction of the normal waves, one must take into account the integral terms omitted in (7).

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Note: Figure translations are in progress. See original paper for figures.

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