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Abstract

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ON THE DISTRIBUTION OF VALUES OF ADDITIVE ARITHMETIC FUNCTIONS

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One of the basic problems in the theory of the distribution of values of arithmetic functions may be formulated as follows: given an arithmetic function $g(m)$; it is required to indicate conditions under which there exist quantities A_n and B_n such that

$$\lim_{n\to\infty} \frac{1}{n} N_n \left\{ \frac{g(m) - A_n}{B_n} \leq x \right\} = F(x), \tag{1}$$

where $F(x)$ is a distribution function; $N_n\{...\}$ denotes the number of natural numbers $\leq n$ satisfying the condition written in braces. If $g(m)$ is a multiplicative or additive function, then it is natural to formulate these conditions in the form of restrictions imposed on its values at powers of prime numbers.

In what follows it is assumed that $g(m)$ is an additive function, $g_1(p) = g(p)$ if $|g(p)| \leq 1$; $g_1(p) = 1$ if $|g(p)| > 1$. In the case of convergence of the series $\sum g_1^2(p)/p$, the problem was solved by P. Erdős and A. Wintner ([5]). For functions for which this series diverges, the most general result belongs to I. P. Kubilius ([2]), who found necessary and sufficient conditions for the existence of the limit (1) and for the convergence of the variances to the variance of the limiting law for functions of the class H introduced by him (with a special choice of A_n and B_n). The class H is characterized as follows: $g(m) \in H$, if

$$\sum_{\sqrt{n}<p^\alpha\leq n} \frac{g^2(p^\alpha)}{p^\alpha} = o\left(\sum_{p^\alpha \leq n} \frac{g^2(p^\alpha)}{p^\alpha} \right)$$

In the present note a proof is given of a necessary and sufficient condition for the existence of A_n and B_n satisfying condition (1), for another class of additive functions, which is not contained in the class H and does not contain it (see also ([3])). This class consists of additive functions which are measurable in a certain sense.

We shall call an additive function measurable if there exists a distribution function $\Phi(x)$ such that, at every point of its continuity,

$$\lim_{n\to\infty} \frac{1}{\pi(n)} \sum_{\substack{p\le n\\ g(p)\le x}} 1 = \Phi(x). \tag{2}$$

From this definition, in particular, it is clear that measurability is connected with the values of $g(m)$ only at prime numbers. For any prescribed distribution function $\Phi(x)$ one can construct an additive function $g(m)$ satisfying condition (2). Such, for example, is the function $g(m)$ for which $g(p) = \inf_{\Phi(x)\ge\{ap\}} x$, if a is any irrational number.

We shall need several lemmas connected with estimates of sums of complex multiplicative functions, which below are denoted by $f(m)$, $f_\nu(m)$.

Lemma 1. Define $\Lambda_f(n)$ by the condition

$$\sum_{d|n} f(d)\Lambda_f\left(\frac{n}{d}\right) = f(n)\log n. \tag{3}$$

Then, if

$$\sum_{n\le x} \Lambda_f(n) \sim \tau\log x; \qquad \sum_{n\le x} |f(n)| \ll \log^A x; \qquad A \ge \operatorname{Re}\tau \ge \eta > 0;$$

$x^\delta \ll y \le x$; δ and η are arbitrary fixed constants; $m(x) = \sum_{n\le x} f(n)$, then

$$m(y) = m(x)\left(\frac{\log y}{\log x}\right)^\tau + o(\log^A x). \tag{4}$$

Proof. Summing (3) over $n \le u$ and using the conditions of the lemma, we obtain

$$\frac{m(u)}{u\log^{\tau+1} u} - \frac{\tau+1}{u\log^{\tau+2} u}\int_1^u m(v)\frac{dv}{v} = O\left(\frac{(\log u)^{A-\operatorname{Re}\tau-1}}{u}\right).$$

Integrating with respect to u from y to x, we obtain (4).

The following two lemmas are proved by a slight modification of the corresponding lemmas of Wirsing ([6]).

Lemma 2. Let

$$\sum_{n\le x} \Lambda_{f_\nu}(n) \sim \tau_\nu\log x; \qquad \sum_{n\le x} |f_\nu(n)| \ll \log^A x; \qquad A_\nu \ge$$

$$\ge \operatorname{Re}\tau_\nu \ge \eta > 0 \quad (\nu = 1, 2, \dots, r); \quad f(n) = f_1(n) * f_2(n) * \dots * f_r(n)$$

($*$ is the operation of multiplicative convolution). Then

$$m(x)=\sum_{n\le x} f(n)=\frac{\Gamma(1+\tau_1)\cdots\Gamma(1+\tau_r)}{\Gamma(1+\tau_1+\cdots+\tau_r)}m_1(x)\cdots m_r(x)+O\left((\log x)^{A_1+\cdots+A_r}\right). \tag{5}$$

Lemma 3. Let $f_1(n)$ be a multiplicative function, $f_1(n)=0$ for even n, $|f_1(p^k)|\ll 2^k$ uniformly in p,

$$\sum_{p\le x}\frac{|f_1(p)|}{p}\log p\sim \tau_1\log x.$$

Then

$$\prod_{p\le x}\left(1+\sum_{k=1}^{\infty}\frac{f_1(p^k)}{p^k}\right)-\sum_{n\le x}\frac{f_1(n)}{n}=O\left(\tau_1^2\prod_{p\le x}\left(1+\sum_{k=1}^{\infty}\frac{|f_1(p^k)|}{p^k}\right)\right). \tag{6}$$

Lemma 4. Let $f(n)=0$ for even n and $|f(n)|\le 1/n$ for all n. Let, further, $f_r(n)$ be the multiplicative function defined by the condition

$$f_r(p^k)=\sum_{m=1}^{k}\binom{1/r}{m}\sum_{\substack{k_1+\cdots+k_m=k\\ k_i>0}} f(p^{k_1})\cdots f(p^{k_m}),$$

i.e.

$$\underbrace{f_r(n)*\cdots*f_r(n)}_{r}=f(n).$$

Then

$$\prod_{p\le x}\left(1+\sum_{k=1}^{\infty}|f_r(p^k)|\right)\ll(\log x)^{1/r}. \tag{7}$$

Proof. For $r=1$ the assertion is obvious. Let $r>1$, $\tilde f(n)=(-1)^{\nu(n)}|f(n)|$. Then

$$|f_r(p^k)|=\sum_{m=1}^{r}\left|\binom{1/r}{m}\right|\sum_{\substack{k_1+\cdots+k_m=k\\ k_i>0}}|f(p^{k_1})\dots f(p^{k_m})|=-\tilde f_r(p^k),$$

$$1+\sum_{k=1}^{\infty}|f_r(p^k)| \leq 1-\sum_{k=1}^{\infty}\tilde{f}_r(p^k) =$$

$$= 2-\left(1+\sum_{k=1}^{\infty}\tilde{f}_r(p^k)\right) = 2-\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)\right)^{1/r} =$$

$$= 2-\left(1-\sum_{k=1}^{\infty}|f(p^k)|\right)^{1/r} \leq 1+\left(1-\frac{1}{p-1}\right)^{1/r} = 1+\frac{1}{rp}+O\Big(\frac{1}{p^2}\Big).$$

This implies (7).

Lemma 5 (Wirsing [6]). Let $|f(n)| \leq 1$, $\sum_{p\leq x} f(p) \sim \tau\dfrac{x}{\log x}$. Then

$$\sum_{n\leq x} f(n) = \frac{\tau x}{\log x}\sum_{n\leq x}\frac{f(n)}{n} + o(x).$$

From Lemmas 1-5 the following follows.

Theorem 1. Let $f(n)$ be a multiplicative function depending on the parameter τ, $|f(n)| = 1$ for all n, and

$$\sum_{p\leq x} f(p) \sim \tau\frac{x}{\log x} \tag{8}$$

uniformly in τ in the region $\operatorname{Re}\tau \geq \delta > 0$ for any fixed δ. Then, uniformly in τ in the same region,

$$\sum_{n\leq x} f(n) = \frac{e^{-C\tau}}{\Gamma(\tau)}\frac{x}{\log x}\prod_{p\leq x}\left(1+\sum_{k=1}^{\infty}\frac{f(p^k)}{p^k}\right) + o(x). \tag{9}$$

Let us apply Theorem 1 to the study of the distribution of values of measurable additive functions for which the series $\sum_p \dfrac{g_1^2(p)}{p}$ diverges. Passing to characteristic functions reduces the problem to an asymptotic estimate of sums of multiplicative functions satisfying the conditions of Theorem 1. From Theorem 1 it follows that if $g(m)$ is a measurable additive function, then, as $n\to\infty$,

$$\frac{1}{n}\sum_{m=1}^{n}\exp i\xi\,\frac{g(m)-A_n}{B_n} = \prod_{p\leq n}\left(1+\frac{e^{i\xi g(p)/B_n}-1}{p}\right) + o(1), \tag{10}$$

if $B_n \to \infty$. The main term of the right-hand side is the normalized and centered characteristic function of the sum of independent random variables ξ_p $(p \leq n)$, where ξ_p takes the value $g(p)$ with probability $1/p$ and 0 with probability $1-1/p$. Using the known limit theorems for sums of independent random variables (see, for example, [1], p. 165; [4], p. 334), we obtain the following assertion.

Theorem 2. Let $g(m)$ be an additive function measurable in the sense of (2), and suppose that the series $\sum_p \dfrac{g_1^2(p)}{p}$ diverges. For the existence of such A_n and B_n that the quantity

$$\frac{1}{n} N_n \left\{ \frac{g(m) - A_n}{B_n} \leq x \right\}$$

converges, as $n \to \infty$, to a proper distribution function, it is necessary and sufficient that there exist a nondecreasing function of bounded variation $L(u)$ such that $L(-\infty) = 0$

and at all points of continuity

$$\lim_{n \to \infty} \sum_{\substack{p \leq n \\ g(p) < u B'_n}} \frac{1}{p} \frac{g^2(p)}{B^2_{n'} + g^2(p)} = L(u).$$

$B'_n > 0$ is determined by the equation

$$\sum_{p \leq n} \frac{1}{p} \frac{g^2(p)}{B^2_{n'} + g^2(p)} = L(+\infty).$$

If this condition is fulfilled, then A_n and B_n can be chosen so that:

$$B_n = B'_n + o(B'_n),$$

$$A_n = \sum_{\substack{p \leq n \\ |g(p)| < B_n}} \frac{g(p)}{p} + B_n \left\{ \int_{|u| \geq 1} \frac{1}{u} \, dL(u) - \int_{|u| < 1} u \, dL(u) - a \right\} + o(B_n),$$

where a is any real number. The logarithm of the characteristic function of the limiting law is computed by the Lévy-Khintchine formula

$$\log \varphi(\xi) = i\xi a + \int_{-\infty}^{\infty} \left(e^{i\xi u} - 1 - \frac{i\xi u}{1+u^2} \right) \frac{1+u^2}{u^2} \, dL(u).$$

Thus, necessary and sufficient conditions have been found under which the existence of a limiting distribution for $g(p)$, with the corresponding normalization and centering, implies the existence of a limiting distribution for $g(m)$ in the case when $g(m)$ is an additive function.

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REFERENCES

1. B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Moscow–Leningrad, 1949.

2. I. P. Kubilius, *Probabilistic Methods in Number Theory*, Vilnius, 1962.

3. B. V. Levin, A. S. Fainleib, Reports of the Academy of Sciences of the Uzbek SSR, 11, 5 (1965).

4. M. Loève, *Probability Theory*, IL, 1962.

5. P. Erdős, A. Wintner, *Am. J. Math.*, 61, 713 (1939).

6. E. Wirsing, *Math. Ann.*, 143, 75 (1961).

Note: Figure translations are in progress. See original paper for figures.