

# ON A REGION FREE OF RESONANCE POLES IN THE SCATTERING PROBLEM FOR A THREE-DIMENSIONAL POTENTIAL

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**Abstract**

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*MATHEMATICAL PHYSICS*

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## ON A REGION FREE OF RESONANCE POLES IN THE SCATTERING PROBLEM FOR A THREE-DIMENSIONAL POTENTIAL

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In the study of the scattering problem for a spherically symmetric potential one is led to the study of the equation

$$u'' + (k^2 - V(r))u = 0, \quad (1)$$

assuming that the potential satisfies

$$\int_0^\infty r |V(r)| dr < \infty. \quad (2)$$

An important question is the distribution of the complex eigenvalues of equation (1). These values correspond to quasistationary states of the particle. The constant  $k^2$  is proportional to the energy of the particle. The theory of quasistationary states is considered in papers (1-3). In papers (4), under the assumption that the potential is finite, it is proved that the poles of the scattering matrix (which are also complex eigenvalues) for the one-dimensional equation (1) at high energies  $k^2 \gg 1$  are located in the lower half-plane of the complex variable  $k = \sigma + i\tau$  under the curve  $\tau = -\ln|\sigma|$ . In the present paper an analogous result is obtained for the three-dimensional Schrödinger equation

$$\Delta u + k^2 u - V(x)u = 0, \quad x = (x_1, x_2, x_3), \quad (3)$$

under the condition that the potential is a differentiable finite function, and it is proved that the wave function  $u(x, k)$  admits analytic continuation into the entire lower half-plane  $\tau < 0$ , with its only singularities being poles. (It is known that for  $\tau > 0$  the function  $u(x, k)$  is analytic in  $k^*$ .) In paper (5) it is proved that, for the one-dimensional equation (1), finiteness of the potential is a necessary and sufficient condition for the existence, for the function  $u(x, k)$ , of

the above-mentioned analytic properties with respect to the variable  $k$ . Questions of analytic continuation of the solution of the Schrödinger equation and of the Green's function for the Schrödinger equation are the subject of papers (6-9).

1°. We shall prove that the solution of equation (3) admits analytic continuation into the half-plane  $\tau < 0$ , and that the only singularities of the analytic continuation are isolated poles (6,7). The solution of equation (3) satisfies the integral equation

$$u = e^{i(\bar{k}, x)} + \int \frac{e^{ik|x-y|}}{-4\pi|x-y|} V(y) u(y, k) dy, \quad (4)$$

\* If the operator (3) has no negative discrete spectrum. In the contrary case the function  $u(x, k)$  is not defined uniquely for those  $k = i\tau$ ,  $\tau > 0$ , for which  $k^2 = -\tau^2$  coincides with one of the points of the discrete spectrum. The function  $u(x, k)$  may be chosen so that it is analytic for  $\tau > 0$ . In what follows, for simplicity, we shall assume that the discrete spectrum is absent.

where the integration is over the entire space,  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3)$  are points of three-dimensional space. It can be verified that the integral operator in (4) depends analytically on  $k$  and, for any complex  $k$ , is a completely continuous operator in the space  $C(E_3, e^{-|x|})$ . We shall now use the following theorem (7, 10), from which the required assertion follows.

**Theorem 1.** *Let the operator  $T(\lambda)$  depend analytically on  $\lambda$ , where  $\lambda \in D$ , and  $D$  is a connected domain in the plane of the complex variable  $\lambda$ . Let  $T(\lambda)$  be a completely continuous operator for each  $\lambda \in D$ . Then either the operator  $I - T(\lambda)$  has no bounded inverse at any point  $\lambda \in D$ , or this inverse exists and is bounded for all  $\lambda \in D$ , except, possibly, for a countable number of isolated points.*

2°. Let us establish that the poles of the function  $u(x, k)$ , located in the half-plane  $\tau < 0$ , for large  $\sigma$  lie below the curve

$$\tau = -a \ln |\sigma| + b, \quad (5)$$

where  $a > 0$ ,  $b$  are constants. For the proof, note that the function  $u(x, k)$  has no singularities in a neighborhood of those points  $k$  for which equation (4) is uniquely solvable. We shall show that for points  $k$  lying above the curve (5), equation (4) is uniquely solvable as  $\sigma \rightarrow \infty$ . This will prove the assertion made above. For the proof let us establish an estimate of the iterated kernel in equation (4):

$$\mathcal{L}(x, y, k) \equiv \int \frac{e^{ik[|x-z|+|z-y|]}}{|x-z| \cdot |z-y|} V(z) dz; \quad z = (z_1, z_2, z_3). \quad (6)$$

The estimate of the kernel  $\mathcal{L}(x, y, k)$  will be carried out according to the scheme proposed in (11). Let us explain the further course of reasoning. We want to show that the kernel  $\mathcal{L}(x, y, k)$  will be small as  $k \rightarrow \infty$ , if  $k$  lies above the curve (5). If this is proved, then, iterating equation (4) once, we arrive at an equation with a small kernel. This equation can be solved by the method of successive approximations, which will complete the proof. We proceed to the estimate of the kernel (6) for large complex  $k$ . We introduce the change of variables used to obtain the estimate for real  $k$  in (11), putting the coordinates of the point  $z$  equal to

$$z_1 = lst + (x_1 - y_1)/2, \quad z_2 = l\sqrt{(s^2 - 1)(1 - t^2)} \cos \psi + (x_2 - y_2)/2, \quad (7)$$

$$z_3 = l\sqrt{(s^2 - 1)(1 - t^2)} \sin \psi + (x_3 - y_3)/2, \quad (8)$$

where  $\psi$  is the angle between the planes containing the vectors  $x - z$  and  $z - y$ , and a fixed plane containing the vectors  $x$  and  $y$ .

The Jacobian of the transformation is equal to

$$J = l^3(s^2 - t^2), \quad (9)$$

$$2l = |x - y|, \quad |x - z| + |z - y| = 2ls, \quad |x - z| - |z - y| = 2lt. \quad (10)$$

The integral (6) is transformed into the form:

$$\mathcal{L} = l \int_0^{2\pi} d\psi \int_{-1}^1 dt \int_1^\infty e^{2ikls} \tilde{V}(s, t, \psi) ds. \quad (11)$$

Denote

$$p(s) = \int_0^{2\pi} d\psi \int_{-1}^1 \tilde{V}(s, t, \psi) dt. \quad (12)$$

Then

$$\mathcal{L} = l \int_1^\infty e^{2ikls} p(s) ds. \quad (13)$$

Together with the potential  $V$ , the function  $p(s)$  is finite and differentiable. We have

$$\mathcal{L} = -\frac{e^{2ikl}}{2ik}p(1) - \frac{1}{2ik} \int_1^\infty e^{2ikls} p'(s) ds \equiv \mathcal{L}_1 + \mathcal{L}_2, \quad (14)$$

where the finiteness of  $p(s)$  was used. Let  $\tau = -\varphi(\sigma)$ ,  $\varphi(\sigma) > 0$ . The first term on the right-hand side of (14), as  $\sigma \rightarrow \infty$ , has order

$$\mathcal{L}_1 = O\left(e^{2l\varphi(\sigma)}/\sqrt{\sigma^2 + \varphi^2(\sigma)}\right), \quad (15)$$

the second term has order

$$\mathcal{L}_2 = o(\mathcal{L}_1). \quad (16)$$

Consequently,

$$\mathcal{L} = O\left(e^{2l\varphi(\sigma)}/\sqrt{\sigma^2 + \varphi^2(\sigma)}\right). \quad (17)$$

Choose the function  $\varphi(\sigma)$  so that, as  $\sigma \rightarrow \infty$ ,

$$\mathcal{L} = o(1). \quad (18)$$

It is clear that in order for (18) to hold, one must set either

$$\varphi(\sigma) = o(\ln |\sigma|), \quad (19)$$

or

$$\varphi(\sigma) \leq c \ln |\sigma|, \quad (20)$$

where  $c > 0$  is a sufficiently small constant depending on  $l$ .

Recall now that  $\tau = -\varphi(\sigma)$ . Inequality (20) allows one to assert that the solution of equation (3) admits an analytic continuation into the domain  $\tau \geq -c \ln |\sigma|$ . Our assertion is proved. It remains to note that the poles of the scattering amplitude

$$f(\mathbf{n}, \vec{\nu}, k) = \int e^{-ik(\mathbf{n}, \mathbf{y})} V(\mathbf{y}) u(\mathbf{y}, k) d\mathbf{y} \quad (21)$$

coincide with the poles of the wave function  $u(\mathbf{y}, k)$ .

Thus, the following has been proved.

**Theorem 2.** *If the potential is a finite differentiable function, then the solution  $u(x, k)$  of Schrödinger's equation (3) admits an analytic continuation to the entire plane of the complex variable  $k$  as a meromorphic function. This continuation is regular in the half-plane  $\text{Im } k \geq 0$ . The poles of the function  $u(x, k)$  lie in the half-plane  $\text{Im } k < 0$  below the curve  $\tau = -a \ln |\sigma| + b$ , where  $a > 0$  and  $b$  are some constants. These assertions remain valid for the scattering amplitude.*

3°. If, instead of finiteness of the potential, one assumes that the condition

$$\int e^{-a|x|} |V(x)| dx < C(a), \quad C(a) = \text{const}, \quad (22)$$

is satisfied, where  $a > 0$  is an arbitrary number, then the function  $u(x, k)$  will admit an analytic continuation to the entire plane of the complex variable  $k$  as a meromorphic function, regular in the half-plane  $\tau > -\tau_0$ , where  $\tau_0 > 0$  is some constant.\* If condition (22) is satisfied for some fixed  $a > 0$ , then analytic continuation is possible, generally speaking, only into the half-plane  $\tau > -a/2$ .\*\* It is meromorphic in this half-plane and regular for  $\tau > -\tau_0$ .\* If the Schrödinger operator is considered in the exterior of a bounded domain with a Lyapunov boundary, and if some self-adjoint boundary condition is imposed on the boundary—

\* See the footnote on p. 1319.

\*\* In works (6,7), in analogous assertions,  $\text{Re } p > -a$  was erroneously written instead of  $\text{Re } p > -a/2$ .

condition, then the solution of the Schrödinger equation admits analytic continuation as a meromorphic function to the whole plane if the potential is finite, or satisfies condition (22), and to the half-plane  $\tau > -a/2$ \* if condition (22) is fulfilled for some fixed  $a$ . In the case of a domain with a boundary, we are unable to indicate a domain of regularity of the analytic continuation in the half-plane  $\tau < 0$ . In the half-plane  $\tau > 0$  the analytic continuation is regular\*\*. In the planar case the assertions made above remain valid after the following changes are introduced. In all assertions the analytic continuation is performed to the plane with a cut along the negative imaginary semiaxis. The assertions made can be applied to estimating the rate at which the solution of a nonstationary problem tends to the limiting amplitude. Let us note here only the following fact. In a planar (or even-dimensional) space the presence of a cut under analytic continuation entails the phenomenon of wave diffusion. If in three-dimensional space, for finite and smooth potentials and initial data, the solution of the problem

$$u_{tt} + \mathcal{L}u = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = g(x), \quad (23)$$

where

$$\mathcal{L}u = -\Delta u + V(x)u \quad (24)$$

decreases exponentially as  $t \rightarrow \infty$ , then in the two-dimensional case, generally speaking, it decreases no faster than  $O(1/t)$ , even for finite and infinitely differentiable functions  $V(x)$  and  $g(x)$ .

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\* See footnote on p. 1321.

\*\* See footnote on p. 1319.

*Note: Figure translations are in progress. See original paper for figures.*

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