A KINETIC EQUATION FOR DESCRIBING MICROPHYSICAL PROCESSES IN CLOUDS

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Abstract

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A KINETIC EQUATION FOR DESCRIBING MICROPHYSICAL PROCESSES IN CLOUDS

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Recently, many publications have appeared in which the kinetic equation has been used to compute the size-distribution function of droplets in clouds. A number of the results obtained are of interest; however, further development of this approach to the phenomenon should proceed not in the direction of developing mathematical methods for solving equations, but along the line of critically reconsidering and discussing the form of the kinetic equation used for such calculations. The point is that the fundamental difficulties in explaining particle growth in the interval $10 \div 20\,\mu$, when gravitational coagulation is still forbidden and condensational growth of droplets is ineffective ([1]), have not yet been removed. Therefore only further study of the elementary processes and investigation of new effects occurring in clouds can lead to the development of our ideas about the formation of the cloud spectrum and can make it possible to refine the very form of the kinetic equation suitable for calculations of real processes of droplet growth. Recently, a number of authors ([2–4]) have developed a theory of stochastic condensation of droplets in clouds. Preliminary estimates show that allowance for this effect substantially increases the rate of condensational growth and makes it possible to explain, in an acceptable way, the real rate of natural precipitation-formation processes. However, the absence of a mathematical apparatus has not yet made it possible to pass from estimates to quantitative calculations with allowance for various meteorological factors.

In the present work, on the basis of results obtained by one of the authors ([2]), a method is developed for calculating stochastic condensation, and a kinetic equation is introduced that describes this process along with others determining the formation of the cloud spectrum. Let us introduce the distribution function of droplets with respect to the squares of their sizes $f(\sigma, \mathbf{r}, t)$, where σ is the square of the particle radius, $\mathbf{r}$ is the radius vector, and t is time.

For an elementary volume, with allowance for the condensational and coagulation growth of particles, the equation is

$$\frac{\partial f}{\partial t} + (u_i - v\delta_{iz})\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial \sigma}\left(\frac{d\sigma}{dt} f\right) = H(f, f) + \varphi, \tag{1}$$

where u_i is the i-th component of the velocity of the medium; v is the sedimentation velocity; δ_{iz} is the Kronecker symbol; φ is a function characterizing the rate of formation of new droplets; $H(f, f)$ is the collision integral, which may be represented in the form

$$H(f,f) = -f(\sigma, \mathbf{r}, t) \int_0^\infty k(\sigma^{3/2}, \sigma {a'}^{3/2}) f(\sigma', \mathbf{r}, t)\, d\sigma' +$$

$$+\frac{1}{2} \int_0^\sigma k(\sigma {a'}^{3/2}, \sigma''^{3/2}, -\sigma {a'}^{3/2}) f\left[(\sigma^{3/2} - \sigma {a'}^{3/2})^{2/3}, \mathbf{r}, t\right] f(\sigma', \mathbf{r}, t)\, d\sigma', \quad (2)$$

where $k(\sigma^{3/2}, \sigma {a'}^{3/2})$ is the coagulation coefficient.

$d\sigma/dt$, in accordance with (2), is determined by an expression of the form

$$d\sigma/dt = Au_z, \quad (3)$$

$$A = c_p(\gamma_a - \gamma'_a)\rho_0 \Big/ 2\pi L \rho \int_0^\infty \sigma^{1/2} f(\sigma, \mathbf{r}, t)\, d\sigma, \quad (4)$$

where ρ_0, ρ are the densities of air and water, respectively; c_p is the heat capacity; L is the heat of condensation; γ_a and γ'_a are the adiabatic and moist-adiabatic temperature gradients.*

In the case of turbulent flow, when the velocity u_i is a random quantity, equation (1) acquires a stochastic meaning and its direct investigation becomes difficult. Separating the systematic and random parts of the function f and performing an averaging operation analogous to that used by Reynolds with the equation of motion, we obtain an equation for mean quantities. Represent u_i and f in the form $u_i = \bar{u}_i + u'_i$, $f = \bar{f} + f'$ and, taking into account that $\overline{\bar{f} f'} = 0$, $\overline{\bar{f} u'} = 0$, $\overline{\bar{u} u'} = 0$, $\overline{f' \bar{u}} = 0$, we obtain

$$\frac{\partial \bar{f}}{\partial t} + (\bar{u}_i - v\delta_{iz}) \frac{\partial \bar{f}}{\partial x_i} + A\bar{u}_z \frac{\partial \bar{f}}{\partial \sigma} + \overline{u'_i \frac{\partial f'}{\partial x_i}} + \overline{Au'_z \frac{\partial f'}{\partial \sigma}} =$$

$$= H(\bar{f}, \bar{f}) + H(f', f') + \varphi + \varphi'. \quad (5)$$

The existence of a dependence of condensational growth on the ascent velocity u_z, generally speaking, leads to the appearance of relations between the variables x and σ, since a change in σ is associated with the displacement of the particle in space. This circumstance leads to the nonconservativeness of the function f, which in turn does not allow one to make the substitution

$$u_i' \frac{\partial f'}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \bar{f}}{\partial x_j}\right),$$

adopted in the case of a conservative substance (here the tensor K_{ij} has the meaning of the coefficient of turbulent diffusion).

Let us generalize the method of passing to a semiempirical equation to the case of turbulent transport of a nonconservative substance. To this end consider the following problem. Suppose that, through a layer of thickness dz, transport of a certain substance occurs due to the motion of the medium and also due to some diffusive mechanism with diffusion coefficient D. For a conservative substance the magnitude of the flux j_z is written in the form $j_z = u_z f - D\,\partial f/\partial z$. If the properties of the substance, specified by some parameter σ, change during displacement in space, then the magnitude of the flux of substance with the given σ will no longer satisfy the above relation for j_z. For it one can easily obtain the equation

$$j_z(z,\sigma) = u_z f(z,\sigma) - D(\partial/\partial z + A\partial/\partial\sigma) f, \quad \text{where } A = \partial\sigma/\partial z.$$

Reasoning analogously, we obtain

$$\partial j_z/\partial z = (\partial/\partial z + A\partial/\partial\sigma)\left[u_z f(z,\sigma) - D(\partial/\partial z + A\partial/\partial\sigma)\right].$$

It is natural to state that, if σ is a function of z, then, by writing this dependence explicitly, one may avoid introducing a second parameter and need not worry about the conservativeness of the substance. The point, however, is that this relation between z and σ is local in character and can be firmly established for small displacements in z. On larger scales, when

* Relations (3), (4) were obtained from the condition of quasi-stationarity, when the change in supersaturation δ follows the change in external parameters. Near the cloud boundaries this approximation is inapplicable; therefore the transition boundary layer of the order of several tens of meters requires separate consideration.

various mechanisms come into play, separately changing f as a function of z and as a function of σ, and, in essence, breaking these local relations; f as a function of only one variable cannot be used. Taking this into account and returning to turbulent transfer, we may write, in the approximation of the semiempirical theory of turbulence, that

$$\overline{f' u_i'} = -K_{ij}\left(\partial/\partial x_j + A_j \partial/\partial\sigma\right)\overline{f}, \tag{6}$$

whence, taking into account that $A_i = A\delta_{iz}$, where A is determined from (4), it is easy to obtain

$$\overline{u'_i \frac{\partial f'}{\partial x_i}} + \overline{A u'_z \frac{\partial f'}{\partial \sigma}} = -\left(\frac{\partial}{\partial x_i} + A\delta_{iz}\frac{\partial}{\partial \sigma}\right) K_{ij} \left(\frac{\partial}{\partial x_j} + A\delta_{iz}\frac{\partial}{\partial \sigma}\right) \overline{f}. \qquad (7)$$

The question of the appearance of new droplets was considered in work ([5]), where expressions for Φ_1 and Φ_2 were also obtained. Finally, we shall simply omit the term $H(f', f')$ in equation (5), since our knowledge is still insufficient to specify the distribution function f' more or less reliably. Apparently, allowance for $\overline{H(f', f')}$ should increase the rate of particle growth. A first step may be the assumption of a normal distribution law.

Thus, finally, omitting the bar sign, we write

$$\frac{\partial f}{\partial t} + (u_i - v\delta_{iz})\frac{\partial f}{\partial x_i} + A u_z \frac{\partial f}{\partial \sigma} = \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial}{\partial x_i} f\right) +$$

$$+\frac{\partial}{\partial x_i}\left(A K_{iz}\frac{\partial f}{\partial \sigma}\right) + A K_{iz}\frac{\partial^2 f}{\partial x_i \partial \sigma} + A^2 K_{zz}\frac{\partial^2 f}{\partial \sigma^2} + \Phi_1 + \Phi_2 + H(f, f). \qquad (8)$$

We shall take this equation as the basis for studying microphysical processes. The equation is nonlinear because of the function $H(f, f)$ and integro-differential, since A and Φ_2 contain the unknown function in the form

$$\int_0^\infty \sigma^{1/2} f(\sigma, \mathbf{r}, t)\, d\sigma.$$

The most important feature of equation (8) is the presence in it of the terms

$$\frac{\partial}{\partial x_i}\left(A K_{iz}\frac{\partial f}{\partial \sigma}\right), \qquad A K_{iz}\frac{\partial^2 f}{\partial x_i \partial \sigma}, \qquad A^2 K_{zz}\frac{\partial^2 f}{\partial \sigma^2},$$

which in fact characterize condensational growth due to pulsations of supersaturation in the cloud. The possibility of taking stochastic condensation into account within the framework of the kinetic equation makes it possible for the first time to begin studying this process with simultaneous allowance for all factors affecting droplet growth, with allowance for inhomogeneity and boundary conditions within the framework of ordinary differential equations. The appearance of additional terms is due to taking into account the fact that the displacement of air volumes by turbulence leads to a change in their physical parameters, and this means that particles are not only mechanically transported by the turbulent flow, but their size also changes. An analytical solution can hardly be found; therefore we shall be able to obtain some results only after a number of simplifying assumptions.

One of the least studied processes of cloud formation is the condensation stage of development, when the droplet sizes are still small and their growth is caused by vapor condensation. In this case, considering a one-dimensional problem (a stratiform-cloud model), we may write

$$\partial f/\partial t + (u - v')\partial f/\partial z + Au\,\partial f/\partial \sigma =$$

$$= K\,\partial^2 f/\partial z^2 + 2AK\,\partial^2 f/\partial z \partial \sigma + A^2 K\,\partial^2 f/\partial \sigma^2 + \Phi_1 + \Phi_2. \qquad (9)$$

Let us consider the change in the distribution function, assuming that the initial distribution depends on z; Φ_1, Φ_2 are equal to zero; A is a constant quantity—and, neglecting the velocity v. Then (9) reduces to

$$\partial f/\partial t + Au\,\partial f/\partial \sigma = A^2 K\,\partial^2 f/\partial \sigma^2, \qquad (10)$$

and its solution for an infinite cloud has the form

$$f(\sigma, t) = \frac{1}{(4\pi A^2 K t)^{1/2}} \int_{-\infty}^{\infty} f(\eta, 0) \exp\left[-\frac{(\sigma - \eta - Aut)^2}{4A^2 K t}\right] d\eta. \qquad (11)$$

If the initial spectrum is monodisperse, then $f(\eta, 0) = Q\delta(\eta - \sigma_0)$, and

$$f(\sigma, t) = \frac{Q}{(4\pi A^2 K t)^{1/2}} \exp\left[-\frac{(\sigma - \sigma_0 - Aut)^2}{4A^2 K t}\right]. \qquad (12)$$

If we pass to the distribution of particles by radius $\psi(r)$, then we shall have

$$\psi(r, t) = \frac{2Qr}{(4\pi A^2 K t)^{1/2}} \exp\left[-\frac{(r^2 - r_0^2 - Aut)^2}{4A^2 K t}\right], \qquad (13)$$

which corresponds to the results obtained from the direct solution of the stochastic equation of particle growth (1).

If the initial distribution depends on z, then the original equation can be written as

$$\partial f/\partial t + u(\partial/\partial z + A\,\partial/\partial \sigma) f = K(\partial/\partial z + A\,\partial/\partial \sigma)^2 f, \qquad (14)$$

the solution of which, when $f(\sigma, z, 0) = Q(z)\delta(\sigma - \sigma_0)$, will be

$$f(\sigma, z, t) = \frac{Q(z - (\sigma - \sigma_0)/A)}{(2\pi A^2 K t)^{1/2}} \exp\left[-\frac{(\sigma - \sigma_0 - Aut)^2}{4A^2 K t}\right]. \qquad (15)$$

If the initial distribution is not monodisperse, but can be represented in the form of the function $Q(z, \sigma_0)$, then $f(\sigma, z, t)$ has the form

$$f(\sigma, z, t) = \int_0^\infty \frac{Q(z - (\sigma - \sigma_0)/A, \sigma_0)}{\sqrt{2\pi A^2 K t}} \exp\left[-\frac{(\sigma - \sigma_0 - Aut)^2}{4A^2 K t}\right] d\sigma_0. \quad (16)$$

Finally, taking into account the appearance of new droplets, when Φ_1 and Φ_2 are not equal to zero, putting

$$Q = Q(\sigma_0, z_0, t_0) = \Phi_1 + \Phi_2,$$

for the case of a stationary source we obtain the solution

$$f(\sigma, z, t) = \int_0^\infty \frac{Q(z - (\sigma - \sigma_0)/A, \sigma_0)}{(2\pi A^2 K t)^{1/2}} \exp\left[\frac{u(\sigma - \sigma_0)}{2AK}\right] \int_0^t \tau^{-1/2} e^{-\beta/\tau - \gamma\tau}\, d\tau\, d\sigma_0, \quad (17)$$

where $\beta = (\sigma - \sigma_0)^2/4A^2K;\ \gamma = u^2/K$.

The relations obtained give the simplest forms of the solution of (9) under severe restrictions imposed on the parameters of the problem. However, even from these solutions one can obtain a number of qualitative results which clearly indicate that the fundamental difficulties in explaining the real growth of particles in clouds are removed as a result of taking into account the stochastic character of condensational growth.

Calculations that more closely correspond to the actual processes require, first of all, abandoning the constancy of A, which leads to an integro-differential equation whose solution can hardly be obtained analytically. The development of numerical methods and the calculation of some cloud models constitute the subject of our further investigation.

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Note: Figure translations are in progress. See original paper for figures.