# ON LINEAR EQUATIONS IN SPACES OF TEST AND GENERALIZED FUNCTIONS

MATHEMATICS

1966

SovietRxiv

## Abstract

## Full Text

UDC 517.941

**MATHEMATICS**

### Z. PRESdorf

# ON LINEAR EQUATIONS IN SPACES OF TEST AND GENERALIZED FUNCTIONS

*(Presented by Academician V. I. Smirnov, 24 V 1965)*

**1°.** Let $\Psi_1, \Psi_2$ be countably normed (c.-n.) spaces*; $\Psi_1', \Psi_2'$ their conjugates; $A$ a linear continuous** operator mapping $\Psi_1$ into $\Psi_2$. We shall use the following notation: $\mathfrak{R}(A)$ is the set of values of the operator $A$; $N(A)$ is the subspace of solutions of the equation $Ax = 0$; $\alpha(A) = \dim N(A)$; $\beta(A) = \dim \Psi_2/\mathfrak{R}(A)$.

The operator $A$ is called **normally solvable** if the equation $Ax = y$ is solvable if and only if $\varphi(y) = 0$ for all $\varphi \in N(A^*)$, where $A^*$ is the operator conjugate to $A$. The property of normal solvability of the operator $A$ is equivalent ($^2$) to the closedness of the set $\mathfrak{R}(A)$. Following ($^3$), a normally solvable operator $A$ will be called a $\Phi_+$-($\Phi_-$)-operator if the number $\alpha(A)$ ($\beta(A)$) is finite, and a $\Phi$-operator if both numbers $\alpha(A), \beta(A)$ are finite.

**Theorem 1.** *In order that $A$ be a $\Phi_+$-operator, it is necessary and sufficient that there exist a continuous operator $B$ mapping $\Psi_2$ into some c.-n. space $\Psi_3$ and such that $BA$ is a $\Phi_+$-operator.*

**Proof.** The necessity is obvious. One may put $B = I$, where $I$ is the identity operator of the space $\Psi_2$.

**Sufficiency.** Denote $BA = S$. Clearly, $\alpha(A) \leq \alpha(S) < \infty$. Let

$$\|\varphi\|_1^{(i)} \leq \dots \leq \|\varphi\|_p^{(i)} \leq \cdots$$

be a countable system of norms of the space $\Psi_i$ ($i = 1, 2, 3$). Decompose $\Psi_1$ into the direct (topological) sum

$$\Psi_1 = N(A) \dotplus N_1 \dotplus \Psi_0,$$

where $N_1$ is a direct complement of the subspace $N(A)$ to $N(S)$. Define the operator $A_1$ as the restriction of the operator $A$ to the subspace*** $\Psi_0 \dotplus N_1$, and prove that the operator $A_1^{-1}$, defined on $\mathfrak{R}(A)$, is bounded.

Let

$$\mathfrak{R}(A) \supset F = \{f \in \mathfrak{R}(A) : \ \|f\|_p^{(2)} \leq C_p \ (p = 1, 2, \ldots)\}$$

be a bounded set. For any element $f \in F$ we have $Bf = SA_1^{-1}f$, $A_1^{-1}f = \xi_f + \eta_f$, where $\xi_f \in \Psi_0$, $\eta_f \in N_1$. The restriction $S_1$ of the operator $S$ to $\Psi_0$ maps one-to-one the c.-n. space $\Psi_0$ onto the c.-n. space $\mathfrak{R}(S)$. Hence (see ($^1$), § 7), the operator $S_1^{-1}$ is bounded, and the set

$$\{\xi_f = S_1^{-1}Bf,\ f \in F\}$$

is bounded in $\Psi_1$.

Let $r = \dim N_1$ (we may assume that $r \geq 1$) and let $x_1, x_2, \dots, x_r$ be a basis of the subspace $N_1$. For any $f \in F$ we have

$$\eta_f = \sum_{i=1}^{r} a_i^f x_i,$$

where $a_i^f$ are constants depending on $f \in F$. We shall prove that $|a_i^f| \leq C =$ const $(i = 1, \dots, r)$, where $C$ does not depend on $f$. Fix an arbitrary number $p_0 \geq 1$. The elements $y_i = A_1 x_i \in \Psi_2 \subset \Psi_{2p_0}$ (see ($^1$)) are linearly independent, and therefore (4) for some $M > 0$

$$\sum_{i=1}^{r} |a_i^f| \leq M \left\| \sum_{i=1}^{r} a_i^f y_i \right\|_{p_0}^{(r)} = M \|f - A_1 \xi_f\|_{p_0}^{(r)} \leq M(C_{p_0} + C'_{p_0}) = C.$$

Hence we obtain $\|\eta_f\|_p^{(1)} \leq C''_p$ for any $p$, i.e., the set $A_1^{-1}F \subset \Psi_1$ is bounded. Thus the operator $A_1^{-1}$ is bounded and, consequently, continuous. It follows that the set $\mathfrak{R}(A_1) = \mathfrak{R}(A)$ is closed.

**Remark 1.** In the particular case when $\Psi_1, \Psi_2$ are Banach spaces, $\Psi_3 = \Psi_1$, and $BA = I + T$ ($T$ is a completely continuous operator), Theorem 1 was proved earlier by S. G. Mikhlin ($^5$).

From Theorem 1 one easily obtains the following result.

**Theorem 2.** *In order that $A$ be a $\Phi$-operator, it is necessary and sufficient that there exist a continuous operator $B$ such that $\alpha(B)$ is finite and $BA$ is a $\Phi$-operator.*

**Remark 2.** Another characterization of the set of $\Phi$-operators is given in Theorem 12 of the paper ($^2$). From the proof of necessity in that theorem it

---

* By a countably normed space we shall mean a complete countably normed space.

** In a c.-n. space the notions of continuity and boundedness of an operator coincide ($^1$).

*** By a subspace we shall mean a closed linear submanifold.

follows that if $A$ is a $\Phi$-operator from $\Psi_1$ into $\Psi_2$, then there exists a continuous operator $B$ from $\Psi_2$ into $\Psi_1$ such that $\alpha(B)$ is finite and $BA = I - L$, where $L$ is a finite-dimensional operator in $\Psi_1$. The converse assertion is true by virtue of Theorem 2.

2°. Let now $\Psi$ be some basic space (in the sense of ($^1$), Ch. II, § 1) such that it is a c.n. space and every function $\varphi(x) \in \Psi$ determines a certain regular functional on $\Psi$, in such a way that different functions correspond to different functionals. Examples of such spaces are the space $C^\infty$, considered in 3°, and also any space of type $K\{M_p\}$ ($^1$) possessing the property: for some $p$ the function $M_p^{-2}(x)$ is summable.

Let $A$ be a linear continuous operator acting in the space $\Psi$ and such that $A^*\varphi \in \Psi$ for any function $\varphi \in \Psi$. Elements $\varphi \in \Psi$ are called **basic functions**, and elements $f \in \Psi'$ are **generalized functions** (g.f.) (over the space $\Psi$). Let $f \in \Psi'$. We shall say that $u \in \Psi'$ is a **generalized solution** of the equation

$$Au = f, \tag{1}$$

if $(u, A^*\psi) = (f, \psi)$ for any function $\psi \in \Psi$. If $f \in \Psi$ and $u \in \Psi$ satisfies equation (1), then the function $u$ will be called a **classical solution** of equation (1). For convenience we shall denote by $A_0^*$ the operator $A^*$ considered on the basic space $\Psi$.

**Definition.** The operator $A$ will be called **normally solvable** in g.f. if equation (1) is solvable (in g.f.) if and only if $(f, \varphi) = 0$ for all $\varphi \in N(A_0^*)$.

**Theorem 3.** *If $A_0^*$ is a $\Phi$-operator, then the operator $A$ is normally solvable in g.f.*

**Proof.** Let $\{\varphi_j\}_{j=1}^s$ be a basis of the subspace $N(A_0^*)$; let $\{y_k\}_{k=1}^q$ be a complete system of generalized solutions of the homogeneous equation (1) (the numbers $s, q$ are finite by assumption). Introduce into consideration the subspace $\Psi_0 = \mathfrak{R}(A_0^*)$. By the condition of Theorem 3, $\varphi \in \Psi_0$ if and only if $(y_k, \varphi) = 0$ ($k = 1, \dots, q$). Construct elements $\xi_i \in \Psi$ ($i = 1, \dots, q$), biorthogonal to the g.f. $y_k$, and let $\Psi_1$ be the subspace with basis $\xi_1, \dots, \xi_q$. Then for the space $\Psi$ we have the representation $\Psi = \Psi_0 + \Psi_1$. Denote by $P_0$ the (continuous) projector of the space $\Psi$ onto $\Psi_0$.

Suppose the conditions are satisfied: $(f, \varphi_j) = 0$ ($j = 1, \dots, s$). Construct the functional $u_0$, putting* for each $\varphi \in \Psi$

---

* We apply the well-known device (see ($^6$)) for constructing a particular solution of an equation of the form (1).

$$(u_0, \varphi) = (f, \varphi^{(0)}). \tag{2}$$

Here $\varphi^{(0)} \in \Psi$ is a solution of the equation $A_0^* \psi = P_0 \varphi$. Let us prove that $u_0$ is a uniquely determined linear continuous functional on $\Psi$. Indeed, let $A_1^*$ be the restriction of the operator $A_0^*$ to the direct complement of the subspace $N(A_0^*)$ in the space $\Psi$. The operator $B = (A_1^*)^{-1}$ is continuous and

$$\varphi^{(0)} = BP_0 \varphi + \sum_{j=1}^{s} a_j \varphi_j.$$

Consequently, $(u_0, \varphi) = (f, BP_0 \varphi)$. It is easy to see that $u_0$ is a solution of equation (1).

3°. We now apply the results from 1° and 2° to the investigation of a system of singular integral equations with Cauchy kernel

$$\mathfrak{A}\varphi \equiv A(t)\varphi(t) + B(t)S\varphi + \int_\Gamma K(t,\tau)\varphi(\tau)\,d\tau = f(t) \tag{3}$$

(where by $S$ we shall denote the singular operator

$$\frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - t},$$

the symbolic matrices $C(t) = A(t) + B(t)$, $D(t) = A(t) - B(t)$ of which degenerate at a finite number of points of the contour $\Gamma$. As the basic space we take the space $C^\infty$ of functions given and infinitely differentiable on $\Gamma$.* We shall assume that $A(t), B(t) \in C^\infty$, $K(t,\tau)$ is infinitely differentiable on $\Gamma \times \Gamma$, and $\Gamma$ is a closed smooth, infinitely differentiable contour bounding a finite simply connected domain.

It is known ($^{7-9}$) that in the spaces $H$ (see ($^7$)) and $L_2(\Gamma)$ the Noether theorems are valid if and only if $\Delta_1(t) \equiv \det D(t) \neq 0$, $\Delta_2(t) \equiv \det C(t) \neq 0$ on $\Gamma$. Nevertheless, the following is valid.

**Theorem 4.** *If each of the functions* $\det(A - B)$, $\det(A + B)$ *has no more than a finite number of zeros of integral multiplicity on the contour, then for equation (3) the Noether theorems are valid in the space* $C^\infty$.

**Proof.** It is clear that the operator $\mathfrak{A}$ maps $C^\infty$ continuously into itself. For simplicity of notation we shall assume that $\Delta_l(t)$ $(l = 1, 2)$ has one root $t = \alpha_l$ of multiplicity $m_l$. Consider the operator $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, where**

$$\mathfrak{B}_l \equiv \frac{I - P_l}{2\Delta_l} H_l [I + (-1)^{lS}], \qquad P_l \psi = \sum_{\nu=0}^{m_l - 1} \frac{1}{\nu!} \psi^{(\nu)}(\alpha_l)(t - \alpha_l)^\nu.$$

Here $H_1, H_2$ are matrices adjugate to $D, C$, respectively; the operators $\mathfrak{B}_1, \mathfrak{B}_2$ are closed and defined on all of $C^\infty$, hence ($^{10}$) they are continuous. It is easy to see that

$$\mathfrak{B}\mathfrak{A}\varphi = \varphi(t) + \int_\Gamma T(t,\tau)\varphi(\tau)\,d\tau,$$

where $T(t,\tau)$ is a matrix of the type $K(t,\tau)$. $\mathfrak{B}\mathfrak{A}$ is a $\Phi$-operator (with index equal to zero) in the space $\widetilde{H}$ (see ($^7$)) and, consequently, in the space $C^\infty$, since every solution of the equation $\mathfrak{B}\mathfrak{A}\varphi = \mathfrak{B}f$ ($f \in C^\infty$) from the class $\widetilde{H}$ belongs to $C^\infty$ and $N((\mathfrak{B}\mathfrak{A})^*) \subset C^\infty$. The conditions of Theorem 2 are fulfilled, and therefore $\mathfrak{A}$ is a $\Phi$-operator in the space $C^\infty$.

From what was said above, on the basis of Theorem 2 and remark ($^{11}$), we obtain the following.

**Corollary.** *If the conditions of Theorem 4 are fulfilled, then the operator* $\mathfrak{A}$ *admits a bounded regularization in the space* $C^\infty$*.*

**Theorem 5.** *If the conditions of Theorem 4 are fulfilled, then the operator* $\mathfrak{A}$ *is normally solvable in o.f. over the space* $C^\infty$*.*

---

* We introduce the topology in $C^\infty$ in the usual way:

$$\|\varphi\|_p = \max_{\substack{0\le k\le p \\ t\in\Gamma}} |\varphi^{(k)}(t)| \qquad (p = 0, 1, \ldots).$$

** In the case of several zeros of the function $\Delta_l(t)$ ($l = 1, 2$), $P_l$ is replaced by the corresponding Hermite interpolation polynomial.

**Proof.** It is easy to see that $\mathfrak{A}^*\varphi = \mathfrak{M}_1\mathfrak{A}'\mathfrak{M}_2\varphi$ for any $\varphi \in C^\infty$, where $\mathfrak{M}_1, \mathfrak{M}_2$ are operators that map $C^\infty$ onto itself one-to-one and continuously in both directions, and $\mathfrak{A}'$ is an operator of the form (3) with coefficients $A'(t), -B'(t)$ (the prime denotes the transposition operation). Thus, $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}'$, and together with them (see (2), Satz 13) also the operator $\mathfrak{A}_0^* = \mathfrak{A}^*$, considered on $C^\infty$, are $\Phi$-operators. Consequently, the condition of Theorem 3 is satisfied.

The author expresses his sincere gratitude to Prof. S. G. Mikhlin for his attention to the present work.

Leningrad State University
named after A. A. Zhdanov

Received
19 V 1965

## REFERENCES

1. I. M. Gel' fand, G. E. Shilov, *Spaces of Test and Generalized Functions*, Fizmatgiz, Moscow, 1958.

2. H. Schaefer, Math. Zs., **66**, No. 2, 147 (1956).

3. I. Ts. Gokhberg, M. G. Krein, UMN, **12**, No. 2, 43 (1957).

4. F. Riesz, UMN, vol. 1, 175 (1936).
5. S. G. Mikhlin, DAN, **125**, No. 4, 737 (1959).
6. M. I. Vishik, S. L. Sobolev, DAN, **111**, No. 3, 521 (1956).
7. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
8. S. G. Mikhlin, UMN, **3**, No. 3 (25), 29 (1948).
9. I. Ts. Gokhberg, UMN, **7**, No. 2, 149 (1952).
10. N. Bourbaki, *Topological Vector Spaces*, IL, 1959.
11. H. Schaefer, Studia math., **18**, No. 3, 229 (1959).

*Note: Figure translations are in progress. See original paper for figures.*