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HYDROMECHANICS

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Abstract

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HYDROMECHANICS

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A NUMERICAL METHOD FOR SOLVING THE EQUATIONS OF HYDRODYNAMICS FOR A PLANE FLOW

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1. A numerical method is described for solving the fourth-order equation

$$\left(-\frac{\partial\Psi}{\partial y}\right)\frac{\partial}{\partial x}\Delta\Psi+\left(\frac{\partial\Psi}{\partial x}\right)\frac{\partial}{\partial y}\Delta\Psi=\frac{1}{\mathrm{Re}}\Delta(\Delta\Psi) \tag{1}$$

under certain arbitrary, but time-independent, conditions for the function Ψ on the boundary of the domain of definition. The method has been used to compute the velocity field in the vicinity of a plate washed by a flow of a viscous incompressible fluid. The results of the computation are presented.

2. Let

$$A\Psi=f \tag{2}$$

be the finite-difference analogue of equation (1) and of the conditions for Ψ on the boundary of the computational domain. We consider a process of successive approximations, represented in the general case as

$$a_{11}a_{12}a_{21}a_{22}\Psi^{k+1}=F\Psi^{k}, \tag{3}$$

where $a_{11},a_{12},a_{21},a_{22}$ are triangular matrices; $F\Psi$ is a certain perturbation of the right-hand side of the original equation (2), making it possible to relate the determination of each subsequent approximation for Ψ to the inversion of four triangular matrices.

In deriving a relation of type (3), we have used, in principle, the idea of the "incomplete factorization" of work ([1]). First of all, if we denote $\Delta\Psi=\Phi$ and pass from differential operators to finite-difference ones, then in this way from (1) we arrive at consideration of the system of two difference equations:

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$$A_1\Phi = F_1\Psi + f_1, \tag{4}$$

$$A_2\Psi = f_2 - \Phi, \tag{5}$$

where $F_1\Psi$ is a vector determined by the conditions for $\Delta\Psi$ on the boundary of the domain under consideration, and f_1 and f_2 are known. Multiply both sides of equation (4) by a certain nonsingular matrix Γ_1 and add the vector $B_1\Phi$ to the left and to the right:

$$\Gamma_1 A_1\Phi + B_1\Phi = B_1\Phi + \Gamma_1(F_1\Psi + f_1). \tag{6'}$$

We determine the meaning of the matrices Γ_1 and B_1 by the condition of representability of $\Gamma_1 A_1 + B_1$ in the form $a_{11}a_{12}$

$$a_{11}a_{12}\Phi = B_1\Phi + \Gamma_1(F_1\Psi + f_1), \tag{6''}$$

where a_{11}, a_{12} are two triangular matrices. Quite analogously, from consideration of (5) one may pass to

$$\Gamma_2 A_2\Psi + B_2\Psi = B_2\Psi + \Gamma_2(f_2 - \Phi) \tag{7'}$$

or

$$a_{21}a_{22}\Psi = \Gamma_2(f_2 - \Phi) + B_2\Psi. \tag{7''}$$

If we denote $Z = a_{12}\Phi$, and $a_{22}\Phi = Y$, then $(6'')$ and $(7'')$ are written as a system of four equations. This system has the form

$$\begin{aligned}
&a_{11}Z^{l+1} = B_1\Phi^l + \Gamma_1(F_1\Psi^l + f),\\
&a_{12}\Phi^{l+1} = Z^{l+1},\\
&a_{21}Y^{l+1} = \Gamma_2(f_2 - \Phi^{l+1}) + B_2\Psi^l,\\
&a_{22}\Psi^{l+1} = Y^{l+1}
\end{aligned} \tag{8}$$

and represents a concrete expression of relation (3).

3. The working form of the general computational scheme:

$$Z_{ik}^{l+1} = \gamma_{1ik}\left[c_{1ik}Z_{i+1\,k}^{l+1} + d_{1ik}Z_{i\,k+1}^{l+1} + (F_1\Psi)_{ik}^l + f_{1ik}\right] + (B_1\Phi)_{ik}^l,$$

$$\Phi_{ik}^{l+1} = \gamma_{1ik}\left[a_{1ik}\Phi_{i-1\,k}^{l+1} + b_{1ik}\Phi_{i\,k-1}^{l+1}\right] + Z_{ik}^{l+1},$$

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$$Y_{ik}^{l+1} = \gamma_{2ik}\left[c_{2ik}Y_{i+1\,k}^{l+1} + d_{2ik}Y_{i\,k+1}^{l+1} + f_{2ik} - \Phi_{ik}^{l+1}\right] + (B_2\Psi)_{ik}^{l},$$

$$\Psi_{ik}^{l+1} = \gamma_{2ik}\left[a_{2ik}\Psi_{i-1\,k}^{l+1} + b_{2ik}\Psi_{i\,k-1}^{l+1}\right] + Y_{ik}^{l+1},$$

$$\gamma_{ik} = \left[e_{ik} + \sigma_{ik} - c_{ik}a_{i+1\,k}\gamma_{i+1\,k} - d_{ik}b_{i\,k+1}\gamma_{i\,k+1}\right]^{-1}, \qquad \sigma_{ik} = \theta e_{ik},$$

$$(B\Omega)_{ik} = \gamma_{ik}\left[d_{ik}a_{i\,k+1}\gamma_{i\,k+1}\Omega_{i-1\,k+1} + c_{ik}b_{i+1\,k}\gamma_{i+1\,k}\Omega_{i+1\,k-1} + \sigma_{ik}\Omega_{ik} + (\Sigma\Omega)_{ik}\right].$$

Here i, k are the parameters of a node of the computational mesh; l is the iteration number; a, b, c, d, e are the coefficients of the function in the finite-difference representation of the corresponding differential equation.

The difference equation is constructed according to the model

$$-a_{ik}\Omega_{i-1\,k} - c_{ik}\Omega_{i+1\,k} - b_{ik}\Omega_{i\,k-1} - d_{ik}\Omega_{i\,k+1} + e_{ik}\Omega_{ik} = f_{ik} + (\Sigma\Omega)_{ik},$$

$$i = 1, 2, \ldots, m; \qquad k = 1, 2, \ldots, n,$$

and in this form it may be regarded as the definition of $(\Sigma\Omega)_{ik}$.

4. In the practice of constructing a finite-difference analogue for the equation

$$u\frac{\partial\Phi}{\partial x} + v\frac{\partial\Phi}{\partial y} - \frac{1}{Re}\Delta\Phi = 0$$

the representation of the convective terms in the form of central differences has become firmly established (2–4). Computational experience shows, however, that a solution for the difference scheme obtained in this way can be obtained only in a limited range of variation of the parameter Re.

The difficulties that arise in solving a difference equation constructed on the basis of approximating the convective terms by central differences naturally disappear if, proceeding from separate approximations of the convective and viscous terms, one introduces for the convective terms the so-called one-sided (“upstream”) approximation—while bringing the number of computational points into correspondence with the required accuracy. We used an approximation scheme with second-order accuracy. In particular, for the derivatives $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$, representations of the form

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Fig. 1. Flow pattern around a plate in the form of isolines $\psi = \text{const}$. The dashed line is the pressure distribution $P(x, y = 0.2\delta)$, $\text{Re} = 100$, $l/\delta = 10$

Figure 1: Fig. 1. Flow pattern around a plate in the form of isolines $\psi = \text{const}$. The dashed line is the pressure distribution $P(x, y = 0.2\delta)$, $\text{Re} = 100$, $l/\delta = 10$

Schematic of the plate in the channel with axes x and y

Figure 2: Schematic of the plate in the channel with axes x and y

$$(\partial\Phi/\partial x)_{ik} = \alpha(\Phi_{ik} - \Phi_{i-1\,k}) - \beta(\Phi_{ik} - \Phi_{i-2\,k}), \qquad \text{if } u_{ik} \geq 0,$$

$$(\partial\Phi/\partial x)_{ik} = \tilde{\alpha}(\Phi_{i+1\,k} - \Phi_{ik}) - \tilde{\beta}(\Phi_{i+2\,k} - \Phi_{ik}), \qquad \text{if } u_{ik} < 0;$$

$\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ are certain positive coefficients determined by the mesh steps.

The approximation of the Laplace operator applied to the function Φ was carried out here by the usual five-point scheme

$$(\Delta\Phi)_{ik} = a\Phi_{i-1\,k} + c\Phi_{i+1\,k} + b\Phi_{i\,k-1} + d\Phi_{i\,k+1} - e\Phi_{ik}.$$

5. The question of the relation between the two approximations is resolved by us, for the problem in a finite-difference formulation, by introducing a certain "convergence parameter" θ. Within the framework of the computational scheme used by us, it was established experimentally that values $\theta \geq 0.2$ ensure convergence of the process over the entire reasonable range of variation of the determining quantities, including a very broad range of variation of the mesh size.

Fig. 1. Flow pattern around a plate in the form of isolines $\psi = \text{const}$. The dashed line is the pressure distribution $P(x, y = 0.2\delta)$, $\text{Re} = 100$, $l/\delta = 10$.

6. Let us consider the flow around a plate of finite length by a stream of a viscous incompressible fluid. Let the fluid flow be bounded by two fixed planes separated from one another by a distance 2δ. The plate is placed along the axis of the channel formed by the two planes and, generally speaking, moves relative to the fixed planes with velocity u_0.

We have computed cases corresponding to various regimes of plate motion for values $\text{Re} = u\delta/\nu$ in the interval $1 \div 1000$. A nonuniform mesh was used, with a concentration of computational points in the region of the nose and tail of the plate, i.e., in those regions where the sought function has singularities.

In Fig. 1 graphs of streamlines and of the pressure distribution along the channel are presented.

For the calculation of the pressure field, the equation

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$$\frac{1}{2}\frac{\partial P}{\partial x} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

was used. Denote the finite-difference representation of the right-hand side at the computational point i, k by F_{ik}. The difference equation is

$$P_{ik} = P_{i-1\,k} + (x_i - x_{i-1})(F_{ik} + F_{i-1\,k})$$

approximates the original differential equation on the solution $P(x)$ with order h^2. The computation of P_{ik} is carried out from left to right. The condition that P be constant in the outlet section may be regarded as a condition for sufficient reliability of the results of the calculation of the velocity field in the region under consideration.

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REFERENCES

[1] N. I. Buleev, *Mat. sborn.*, **51**, No. 2 (1960).
[2] L. M. Simuni, *Zhurn. vychisl. matem. i matem. fiz.*, **5**, No. 6 (1965).
[3] I. Yu. Brailovskaya, *DAN*, **160**, No. 5, 1042 (1965).
[4] H. E. Kalis, A. B. Tsinober et al., *Magnitnaya gidrodinamika*, **1** (1965).

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