ON THE THEORY OF BOUNDARY-VALUE PROBLEMS FOR THE EQUATION

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Abstract

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MATHEMATICS

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ON THE THEORY OF BOUNDARY-VALUE PROBLEMS FOR THE EQUATION

$y^m u_{xx} + u_{yy} = k(x, y)e^u$ IN THE LARGE

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Let D be a simply connected domain of the half-plane $y > 0$, bounded by the segment AB of the x-axis and by a curve σ, Hölder-continuous in the sense of a curve, which approaches the x-axis orthogonally at the points $A(-1, 0)$, $B(1, 0)$. Consider the equation

$$Eu(x, y) = y^m u_{xx} + u_{yy} = k(x, y)e^{u(x,y)}, \tag{1}$$

where m is a positive number, and $k(x, y)$ is a function nonnegative everywhere in the domain D, possessing continuous first derivatives in $\overline{D}$. Let $F(s)$ be a continuous function, prescribed on the boundary $\Gamma = \sigma + AB$ of the domain D, of the arc abscissa s, measured in the positive direction.

Theorem 1. *There exists a unique solution, twice continuously differentiable in the domain D, of equation (1), continuous in the closed domain $\overline{D}$ and satisfying the condition*

$$u(x, y)\big|_\Gamma = F(s). \tag{2}$$

In view of the fact that $k(x, y) > 0$ in the domain D, the uniqueness of the solution of problem (1)–(2) follows from the extremum principle ([1]).

We shall carry out the proof of the existence of a solution according to the scheme proposed in ([2]).

The solution $u_0(x, y)$ of the equation

$$Eu(x, y) \equiv 0, \tag{3}$$

satisfying condition (2), exists and is unique. The existence of the Green function for problem (2)–(3) was proved in ([3]). In particular, in the case of the so-called normal domain, when the curve σ is given by the equation

$$x^2 + \frac{4}{(m+2)^2} y^{m+2} = 1, \qquad y \geqslant 0,$$

the function $G(x, y; x_1, y_1)$ is constructed explicitly ([3,4]).

With the aid of the Green function, problem (1)–(2) is reduced to the equivalent integral equation

$$u(x, y) = -\int_D G(x, y; x_1, y_1)\, k(x_1, y_1) e^{u(x_1, y_1)}\, dx_1\, dy_1 + u_0(x, y). \tag{4}$$

Into equation (4) we introduce the parameter λ:

$$u(x, y) = -\lambda \int_D G(x, y; x_1, y_1)\, k(x_1, y_1) e^{u(x_1, y_1)}\, dx_1\, dy_1 + u_0(x, y). \tag{5}$$

A solution $u_\lambda(x, y)$ of equation (5) always exists for small values of the parameter $\lambda > 0$.

If we show the existence of a solution of equation (5) for $\lambda = 1$, then thereby the existence of a solution of problem (1)–(2) will be proved.

We note that, for any $\lambda > 0$, every solution of the integral equation (5) is bounded.

Indeed, by virtue of the inequality $G(x, y; x_1, y_1) > 0$, it follows from (5) that $u_\lambda(x, y) \leq u_0(x, y)$, i.e.

$$0 < e^{u_\lambda(x,y)} \leq e^{u_0(x,y)} \leq e^{\max u_0(x,y)} = e^{\max F},$$

and, consequently,

$$u_0(x, y) - \lambda \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{\max F}\, dx_1\, dy_1 \leq u_\lambda(x, y) \leq u_0(x, y). \tag{6}$$

Suppose that the solution $u_\nu(x, y)$ of the integral equation (5) is known for $\lambda = \nu < 1$. Starting from this, we shall prove the existence of a solution $u_{\nu+\alpha}(x, y)$ of equation (5) for $\lambda = \nu + \alpha$, where α is some positive number.

It is not difficult to see that the function

$$\xi(x, y) = u_{\nu+\alpha}(x, y) - u_\nu(x, y)$$

satisfies the integral equation

$$\begin{aligned}\xi(x,y) &+ (\nu+\alpha)\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\nu(x_1,y_1)}\xi(x_1,y_1)\,dx_1\,dy_1 \\ &= -\alpha\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\nu(x_1,y_1)}\,dx_1\,dy_1 \\ &\quad -(\nu+\alpha)\sum_{n=2}^{\infty}\frac{1}{n!}\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\nu(x_1,y_1)}\xi^n(x_1,y_1)\,dx_1\,dy_1.\end{aligned} \tag{7}$$

By virtue of (6) we have

$$\max|\xi(x,y)| = \delta \le \max\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{\max F}\,dx_1\,dy_1 = d. \tag{8}$$

Since problem (2) with zero right-hand side for the equation

$$E\xi(x,y) = (\nu+\alpha)k(x,y)e^{u_\nu(x,y)}\xi(x,y)$$

is equivalent to the integral equation

$$\xi(x,y) + (\nu+\alpha)\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\nu(x_1,y_1)}\xi(x_1,y_1)\,dx_1\,dy_1 = 0, \tag{9}$$

the latter cannot have nontrivial solutions. Hence, in turn, it follows that equation (7) is equivalent to the integral equation

$$\begin{aligned}\xi(x,y) &= -\alpha\int_D R(x,y;x_1,y_1;\nu+\alpha)\,dx_1\,dy_1 \\ &\quad -(\nu+\alpha)\sum_{n=2}^{\infty}\frac{1}{n!}\int_D R(x,y;x_1,y_1;\nu+\alpha)\xi^n(x_1,y_1)\,dx_1\,dy_1,\end{aligned} \tag{10}$$

where $R(x,y;x_1,y_1;\nu+\alpha)$ is the resolvent of the kernel

$$(\nu+\alpha)G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\nu(x_1,y_1)}.$$

Since the function

$$\int_D R(x,y;x_1,y_1;\nu+\alpha)\,dx_1\,dy_1, \qquad 0 \le \nu+\alpha \le 1,$$

is continuous in the closed domain $(x,y) \in \overline{D}$, the estimates

$$0 \le \int_D R(x,y;x_1,y_1;\nu+\alpha)\,dx_1\,dy_1 \le B, \tag{11}$$

hold, where B is a constant.

From (8) and (11) follows the validity of the inequalities

$$(\nu+\alpha)\left|\sum_{n=2}^{\infty}\frac{1}{n!}\int_D R(x,y;x_1,y_1;\nu+\alpha)\zeta^n(x_1,y_1)\,dx_1\,dy_1\right| \le B_1\delta^2,$$

$$(\nu+\alpha)\left|\sum_{n=2}^{\infty}\frac{1}{n!}\int_D R(x,y;x_1,y_1;\nu+\alpha)\{\zeta^n(x_1,y_1)-\bar{\xi}^n(x_1,y_1)\}\,dx_1\,dy_1\right| \le B_2\max|\zeta(x,y)-\bar{\xi}(x,y)|,$$

where $\bar{\xi}(x,y)$ is an arbitrary function of the class $C(\bar{D})$, $\max|\zeta(x,y)|<d$, and the positive constants B_1 and B_2 do not depend on ν and α.

According to these inequalities, from (10) we obtain

$$\delta \le aB + B_1\delta^2.$$

Let

$$\tau = 1-\sqrt{1-\alpha\cdot 4BB_1}/2B_1$$

be the smaller root of the quadratic equation

$$\tau = aB + B_1\tau^2.$$

In the work ([2]) it is proved that if

$$B_2\tau = q < 1, \tag{12}$$

then there exists a unique solution of the integral equation (10), satisfying the condition $|\zeta(x,y)|<\tau$.

Condition (12) will be satisfied for

$$a < (4B_2 - B_1)/4BB_2^2 = a^*. \tag{13}$$

This proves the existence of the solution $u_{\nu+\alpha}(x,y)$ for any positive $a<a^*$.

Repeating the above argument step by step, we reach the value $\lambda=1$. Otherwise this process would break off for some value of the parameter $\lambda=\mu+\alpha$, i.e. the integral equation

$$\zeta(x,y)+(\mu+\alpha)\int_D G(x,y;x_1,y_1)k(x_1,y_1)e^{u_\mu(x_1,y_1)}\zeta(x_1,y_1)\,dx_1\,dy_1 = 0$$

would have a nontrivial solution, which is impossible.

Thus the existence of a solution of problem (1)–(2) has been proved.

Theorem 2. *There exists a unique twice continuously differentiable in the domain D solution of equation (1), satisfying the conditions*

$$u(x,y)\big|_{\sigma} = \varphi(s), \tag{14}$$

$$\partial u(x,y)/\partial y\big|_{y=0} = \nu(x), \tag{15}$$

where $\varphi(s)$ is a prescribed continuous function on σ, and $\nu(x)$ is also a prescribed continuous function on the interval $0 < x < 1$, which may tend to infinity of order less than $2/(2+m)$ as $x \to 0$ or $x \to 1$.

The uniqueness of the solution of this problem again follows from the extremum principle.

Equation (3) has a unique solution satisfying conditions (14) and (15), which can be written with the help of the corresponding Green' s function $G^{*}(x,y;x_1,y_1)$. For this problem the Green' s function exists, and it is written explicitly in the case of a normal domain ([5,6]).

The problem (1)–(14)–(15), with the aid of the Green' s function, is reduced to a nonlinear integral equation, whose study is carried out analogously to the study of equation (5).

The boundary-value problems (2) and (14)–(15) for the equation

$$Eu(x,y) = f(x,y;u)$$

are studied analogously, under the condition that the function $f(x,y;u)$ can be expanded in a series in powers of $u(x,y)$, is positive everywhere in the domain D, is continuous in x, y, and the derivative $f'_u(x,y;u)$ is nonnegative in the domain D.

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Note: Figure translations are in progress. See original paper for figures.